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ABSTRACT

In this study, Piagetian theory is extended to explore ways in which children construct their understanding of our notational system and place value. Eighty children aged 4 to 9 were asked to group objects, draw pictures of grouped quantities, write numerals to indicate amounts, and theorize about the relationship between their written numerals and drawn quantities. Developmental levels were inferred for these and other tasks, and level x age analyses were performed. Eighteen hypotheses that children brought to bear on the meaning of the notational marks in relation to the symbolized quantities were formed, and were grouped into five developmental levels. Among the patterns that emerged was a developmental sequence in the kinds of ideas used by children, singly and in combination. Children's understanding of place value seems to be built in phases over a long period of time, in conjunction with other kinds of knowledge. Some developmental relations were evident; however, the children appeared to have many theories that intruded upon their understanding of the numeration system. Some parallel findings between the historical development of our system and the developing knowledge of that system in children are pointed out. A selection of the children's graphic productions is appended. (Author/MNS)

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FINAL REPORT

Title: THE DEVELOPMENT OF CHILDREN'S UNDERSTANDING OF NUMERICAL REPRESENTATION

Project Number: NIE-G-80-0094

September 1, 1981

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Marcus Lieberman
Principal Investigator

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Abstract

Number is an idea, and numerals are notational marks used to represent number. In this study Piagetian theory is extended to explore the ways in which children construct their understanding of our notational system and, in particular, of the place value property of the decimal numeration system. Eighty middle-class children ranging from four to nine years of age were interviewed. They were asked among other things to group objects, draw pictures of grouped quantities (symbolic representation), write numerals to indicate amounts (conventional representation), and theorize about the relationship between their written numerals and drawn quantities. Developmental levels were inferred for these and other tasks, and level x age analyses were performed. A selection of the children's graphic productions are included in an appendix.

The central results of this descriptive study consist of eighteen hypotheses that children brought to bear on the meaning of the notational marks in relation to the symbolized quantities. The hypotheses were grouped into five developmental levels, the highest of which reflects a knowledge of place value. Among the patterns that emerged was a developmental sequence in the kinds of ideas used by children, singly and in combination, that may best be described by some form of ordinal data analysis.

Children's understanding of the place value property, rather than being constructed all at one time and in relative isolation from other learning, seems to be built in phases, over a long period of time, in conjunction with other kinds of knowledge. Some developmental relations were evident among children's ability to group and draw objects, write

numerals, and offer hypotheses concerning the relationship between notational marks and numerical quantities. But child-learners appear to have many theories, not directly related to the cognitive capacities that were examined, that intrude upon their understanding of the elements and properties, both numerical and notational, of the numeration system. Some parallel findings between the historical development of our system and the developing knowledge of that system in children are pointed out.

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CHAPTER I

INTRODUCTION

Number is an idea, and numerals are the notational marks we use to represent number. For adults who are familiar with both number and numerals, the relationship between them is straightforward: the latter usually stands for or implies the former. Rarely do we have occasion to ponder the different meanings which numerals themselves can have when they are used in one everyday context rather than another. The numeral 11, for example, can indicate the cardinal value of a collection of objects containing that many elements (e.g., the number of children in a class composed of five boys and six girls, or the quantity of eggs left in an egg carton after one of them has been removed). The numeral 11 can signify the amount of a continuous quantity measured in standard units (e.g., age in years, cost in dollars, time in minutes). Or the numeral 11 can mark position or location when it functions as an ordinal label (e.g., the house standing between No. 9 and No. 13 on a block, or the street falling between Tenth Avenue and Twelfth). In the context of telephone numbers, the numeral 911, with 9 preceding 11, signals "emergency;" 411, with a 4 replacing the 9, conjures "information." Adults readily understand that the graphic mark 11 carries different meanings when it is used in one way rather than another. But for children who have not yet structured the quantities signified by a numeral (especially multi-digit numerals), or for whom the meanings of numerals are not yet differentiated by function or contextual occurrence, the relationship between number and numerals cannot be obvious.

From the perspective of the young child, numerals might simply be marks that are linked to particular objects (a numeral on a tee shirt linked with that particular article of clothing) or events (Boston's Channel "2" logo with viewing Sesame Street). They may be squiggles that are loosely associated with counting words (graphic marks that are treated like objects to which the action of pointing may be applied, accompanied by the enunciation of the string of counting words). They may be arbitrary sequences of marks having no intrinsic rhyme or reason (as in telephone numbers or license plates). They may be notational elements akin to alphabetic letters, by means of which numbers can be "spelled" (twelve is made with a 1 followed by a 2, with no space left in between). They may help to locate events in the qualitative rather than a quantitative way (1776 and 1492 were before I was born but after the prehistorical animals). All of these are reasonable possibilities, but none of them are numerical in the strict sense.

The research described in these pages is an exploration into how children construct their understanding of our conventional system of representing numerical quantities, that is, the decimal or base ten system of notation which uses the digits 0 through 9 and place value. Place value is the idea that, e.g., the digit 1 means one, ten, or one hundred, depending upon its written position relative to the decimal point. Children's understanding of the place value property of the notational system, used to organize the ten digits in a specific way to convey numerosity, is the central concern of this study.

My curiosity in this topic is an outgrowth of having observed that many children have difficulty understanding the place value property of our numeration system. The numeration system is introduced in the first grade and is taught throughout the second and third grades (e.g., Thompson, in preparation; Easley _et al_., 1979; Hatano, 1979; Ginsburg, 1977; Resnick, 1976; Smith, 1973; Rathmell, 1972; Scrivens, 1968). Many second and third graders cannot surmount barriers to place value comprehension, but more surprisingly, the difficulty persists for some children into the fifth and subsequent grades. The problem is well known to math educators and elementary school teachers. They have recognized and struggled with it for years (e.g., Labinowicz, 1980; Lerch _et al_., 1979; Maddell, 1979; Good, 1979; Payne and Rathmell, 1975; Wirtz, 1974; Wheeler, 1971; Churchill, 1961; Van Engen, 1947).

Curriculum designers and teachers naturally want children to have a full understanding of what they are doing when they work in "symbolic arithmetic." Furthermore "meaningful" arithmetic demands an appreciation on the part of the learner of what those symbols stand for (Van Engen, 1947). Thus it is felt that the concept of place value has to be taught before, or at least alongside, the algorithms for the arithmetic operations.

The educators' decision to introduce the numeration system to children in the early grades is predicated on several notions. The first is that notational arithmetic is better than other kinds, that is, paper-and-pencil "symbolic arithmetic" is more abstract and more useful than working with manipulatives (concrete objects) or verbal forms (Baratta-Lorton, cited in Labinowicz, 1980; Wirtz, 1974). The second is that the different uses and meanings which numerals have in everyday life are either not a source of confusion (i.e., children can readily differentiate among meanings and grasp the connections among different uses) or not an issue of concern (i.e., children simply juxtapose them). Third is the notion that there is a direct correspondence between verbal description ("two tens and three ones") and graphic notation ("23"), and that these representational forms can easily be linked with the quantities themselves (23 objects of whatever specific kind). Fourth is the idea that teaching algorithms, or procedures for carrying out the arithmetic operations, is the best way for children to become familiar with (learn) large numbers (Wheeler, 1971). Fifth is the presumption that the logico-arithmetic relations underlying the operations and the numeration system will become evident to the child, once he or she demonstrates the proper use of the learned algorithms.

Recent research in psychology, mathematics education, and artificial intelligence has made it increasingly clear that graphic notations (marks made on two-dimensional surfaces, such as straight or curved lines, dots, geometric figures, letters, numerals) do not carry meaning in themselves. Instead a child constructs meaning from previously acquired knowledge and contextual cues, imposing on these representational devices his/her particular theories and procedures for figuring out what the marks mean on the one hand, and what the marks can be made to convey on the other. Neither the content of this construction, nor the processes by which conventional meanings are learned, are as yet well specified.

The Educational Perspective

Over the years teachers have sought the advice of mathematics educators, mathematicians, and psychologists in their quest for ways to facilitate children's grasping of "the place value concept." The specialists have created an armory of materials which are regularly used by many teachers. A sampling of these includes materials designed to be "concrete embodiments" of the place value idea (e.g., Dienes blocks, Unifix cubes, bean sticks, and base ten abacuses); games that require the players to make exchanges between objects representing small units and objects standing for units of higher value, such that children gain practice in "seeing the equivalence" between many (small values) and one (higher values) (e.g., chip trading and bankers' games); and play money which is intended to link place value lessons with a content that is presumably of intrinsic interest and practical utility to children.

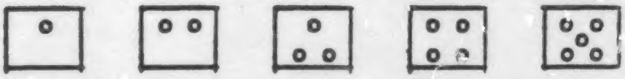
While there is considerable variation in the choice of materials and activities used in particular classrooms, the rationale for employing them typically rests upon several notions regarding children's learning that piggy back on, or mutually support, one another. Four such widely held assumptions are the following:

1. Children's learning proceeds from the concrete to the abstract;
2. Children learn by doing, by acting, by sensing;
3. Children learn new material in a step-by-step fashion;
4. Children need to be motivated in order to learn.

The pedagogical prescription that is linked with these presuppositions is that good teaching ought to be consistent with children's natural learning tendencies. As a whole this conceptualization of the learning process enjoys the confidence of most educators and child development specialists.

Arithmetic textbooks mirror this formulation of children's learning. Typically first grade texts use the following sequence to teach the numbers from one through ten.

1. Pictures of objects (e.g., balloons, apples, insects) are presented in increasingly larger set sizes.



etc.

2. The child is told that each is a number and is called "a set of ___" (the blank is the word for the pictures quantity of objects).
3. The child is taught to write the number five (numeral 5) in the blank space accompanying the picture and to verify his/her answer by counting the objects.
4. The child presumably links the final count word with the pictured quantity of objects on the one hand, and the written numeral on the other.

As this example shows, the step-by-step progression consists of movement, by substitution or association, through two levels of abstraction: a lower pictorial level (objects are replaced by pictures of them); and a higher symbolic level (pictures are replaced by numerals). The substitution of numeral for picture is mediated by the procedure of counting. Number is thus a property of the set of objects (or pictures) and can be abstracted from it via counting.

In the classroom the assumptions regarding children's learning are reflected in the following kinds of ways. Kindergarteners and first graders are encouraged to work with concrete objects because they can bring the whole range of their perceptual machinery to bear upon them. The teachers' tasks are several: to help the children "see" the correspondence between objects, pictures, and numerals, as well as physical actions, operational signs, and other conventional notational devices; to motivate children to learn by arousing their interest through games and other enjoyable activities; and to move the children as expeditiously as possible from a reliance on manipulatives to mastery with paper-and-pencil "symbolic arithmetic." Parenthetically it might be added that after the first or second grade, manipulatives are usually reserved for those occasions when children show that they are having trouble working in symbolic arithmetic, that is, when they produce computational errors which indicate that they are not performing an algorithm in the way in which it _ought_ to be done. Unfortunately manipulatives become "babyish" remedial aids from the perspective of older children.

From the Piagetian point of view, children do not learn number concepts in the way suggested in the foregoing discussion. First, sets of objects do not contain number. Number (e.g., the concept of five-ness, or the _meaning_ of five) is constructed from _within_ the child, and he/she imposes that meaning upon objects, pictures, and notational devices. Pre-structured sets of ten objects (e.g., tens-rods) or pictorial representations (e.g., five ladybugs or five balloons) can serve as vehicles for engaging children in counting and writing and exchanging. But neither the objects nor the pictures of ready-made sets "contain the number to be abstracted."

Second, the signs used for representing number (number words in the verbal system and numerals in the written system) are culturally given, conventional devices. They are present in the environment, transmitted to the child from the outside, and learned by means of imitation, informal and/or direct instruction and practice. But the meanings of these signs (the ideas they stand for) are constructed from within. Counting words, first learned as verbal strings, or as utterances made while pointing to objects, have to be linked with the idea of the numerical quantities they signify; they have to be assimilated into the notion of the cardinality of the set. Similarly, the graphic mark 5, taught from the outside, has to be assimilated into the idea of five, as well as being linked with the counting word five.

In sum, conventional methods of representing numerical ideas have to be taught from the outside, but the ideas themselves have to be constructed from within. The numerals, like the counting words, are learned by imitation and are consolidated with exercise or practice. But the ideas for which they stand are constructed in a different way. The mechanisms of construction of these and other numerical concepts, such as those underlying the numeration system, are in need of better description. My presumption is that there should be a close relationship between what is taught, how it is taught, and children's natural construction of knowledge. This is the stance from which the very general question of how children construct their understanding of the conventional notational system is raised.

The Psychological Perspective

The place value problem is challenging from the standpoint of teaching and curriculum design, but it is intriguing from a psychological point of view as well. Embedded in the problem are numerous questions concerning the relationship between children's development of a general universal cognitive structure (number), and their acquisition of the cultural object for representing number (numeration system). If the locus of children's difficulties is in the construction of certain number concepts, then it would be useful to identify which ones. If children's problems arise in reconstructing the conventional notational system, then it would be helpful to specify which aspects of the reconstruction are problematic and how each might be overcome.

Another set of questions revolves around the coordination of verbal and graphic representational devices for arriving at answers to questions of "how much" or "how many." The lack of direct correspondence between the linguistic terms we use in talking about number and the notational marks we use in recording number has been noted (e.g., Sinclair, 1980). How do children construct the linkages between these separate tools? And how do they coordinate their knowledge of these externally given representational devices with their knowledge of number which, rather than being learned or taught from the outside, is constructed from within?

One approach to these questions — an approach suggested by the work of Piaget — is to find out what ideas the children themselves have about number and numerals which stand in opposition to, or clash with, instruction given by adults in the numeration system. It is possible that children have some powerful notions that they find difficult to coordinate with the place value concept. If we knew more about these ideas, we might be able to shed light on the teaching/learning difficulties that persistently show up in the elementary grades.

The study related in this thesis addresses this possibility. Utilizing Piaget's distinction between representation using symbols (personally constructed graphic marks that resemble the things being represented, as in drawings) and representation using signs (conventional notational marks that, like digits, are removed from the thing that is signified and like numerals, are part of a system of marks), I set out to try to uncover what relationships there were between children's number concepts, their personal representations of quantities of objects (in magic marker and crayon drawings), and their understanding of the digits and numerals that are used to signify the amounts that they had drawn.

The study is conceived in terms of the development of meanings which children impute to the notational system. The central focus of the empirical work is the development of children's ideas regarding the significance of one-, two-, and three-digit numerals. I wanted first to find out what theories children had about the relation of digits and numerals to numerical quantities, and second to see whether those theories were idiosyncratic (i.e., one child, one theory, in random chronological order and without reference to conceptual development) or constituted a developmental sequence among children across ages and/or years of schooling. I felt that studying place value understanding in isolation from the development of other numerical abilities and other representational forms would yield an impoverished description of the development. Therefore I designed a series of tasks, each with a slightly different focus, to provide a richer data base for the interpretation of the central results.

The Piagetian Perspective

The framework for this study is derived from Piaget's genetic epistemological inquiries into the development of scientific concepts in the child as well as in history. Piaget's studies are guided by his interactionist and constructivist viewpoint on the development of knowledge, his structural analysis of action/thought, and his concern with man as a biological creature who needs to adapt to his environment. The Genevan investigative approach uses two methods of study: the historical-critical method which begins with the present and looks backwards in time at the construction of a scientific concept; and the psychogenetic method which focuses on the origins and successive understandings of that concept in the cognitive development of the child. In both cases the main object of study is the way knowledge changes, or how transitions take place from less developed to more developed states (Inhelder, 1962; Berthoud-Papandropoulou and Ackermann-Valladao, 1980).

The general hypothesis of Piaget's approach is that conceptual change in science is a function of man's search for progressively more general, more inclusive laws which will (a) explain sets of occurrences (laws); (b) explain the relations among those laws (i.e., higher order, more general explanations); (c) resolve conflicts generated by competing viewpoints; (d) account for anomolies or exceptions, and so forth (Piaget, 1968). To the extent that children's conceptual development may be characterized by the construction of ever more general, more inclusive, and more powerful relationships which serve to make their understanding more coherent and objective, Piaget's approach informs the study of knowledge building by providing the opportunity to subject hypotheses concerning the development of knowledge to empirical inquiry. This is especially valuable in looking at the origins of a concept (even the most primitive of concepts known to us were products of adult thinking), and when the historical records of discovery are incomplete or nonexistent (Berthoud-Papandropoulou and Ackermann-Valladao, 1980). Likewise understanding the transitions from less adequate to more adequate notions in a particular domain can be a rich source of hypotheses concerning the developmental course of the construction of knowledge in the child.

Piaget's biological concerns are well known (Piaget, 1971; 1963). In his view man, like all other organisms, adapts to his environment. The general processes by which adaptation occurs are described by the notions of assimilation and accommodation. The developmental course of change is conceptualized in terms of successive levels of cognitive organization. But unlike lower organisms, man has a special tool by means of which he extends his adaptive capacities to rise above the limits of his immediate environment. That special tool is called human intelligence: with it man reconstructs the past, reorganizes the present, anticipates the future, and thereby extends his capacities beyond the present to the possible, and beyond the limited plane of action to the wider plane of thought. It is this tool whose origins and functioning Piaget has sought to describe and explain in his numerous studies over the past sixty years.

The Genevan psychogenetic approach takes into account both the structural and the functional aspects of the development of a concept or theory. The structural analysis aims at uncovering the very general underlying structures of knowing that direct a subject's behavior. The functional viewpoint is concerned with the cognitive processes or procedures by which new understandings are made possible. The approach is interdisciplinary in that it relies upon theoretical analyses in particular content areas (e.g., concepts in physics or mathematics) and psychological experimentation with children.

The Genevan framework is used here. Chapter II begins with a theoretical analysis of numeration systems as currently understood by mathematicians, and is followed by a historical survey of major milestones in the development of them. The chapter closes with a review of previous literature in the fields of psychology and mathematics education which bear on the developmental/learning issues addressed in this thesis. Chapter III is made up of two parts. The first part sets forth the theoretical framework for the psychogenetic or experimental portion of the study. The second itemizes the hypotheses which were postulated to study the development of children's understanding of numerical notation. In Chapter IV the methods used in carrying out this study are described. Included there are descriptions of the tasks which were given and the procedures which were followed in administering them; the selection of subjects; the interview format; and data analysis procedures. The empirical findings are reported in Chapter V.

A study of this nature has broader implications. My hope in conducting the interviews was to gather evidence that (a) children hold ideas which are different from those of adults, (b) those notions form a developmental sequence, and (c) the uncovering of those ideas would point the way toward identifying some of the clashes that must occur between children's independently constructed ideas and the instruction they are receiving in school. This study is linked to the important educational issue of facilitating children's understanding of the place value property of the conventional notational system. The clashes might help to explain children's resistance to our instructional endeavors and offer some suggestions as to what can be modified: our expectations of children's capacities for reconstructing the numeration system for themselves; our generally held presuppositions of how children learn in this area; our teaching methods including the materials and activities we want children to use; or our curricular timetable for instructing children in the numeration system. Chapter VII contains a discussion of what this study adds to our knowledge regarding these issues, and perhaps more importantly, what we have yet to uncover. My speculations regarding the merit of extending this inquiry into other areas of mathematics education are included there.

CHAPTER II

BACKGROUND AND LITERATURE REVIEW

Introduction

The intent of this chapter is to summarize the research done in preparation for the study of children's developing understanding of the conventional notational system. The format of the presentation will be familiar to those who are accustomed to reading genetic epistemological research but will seem unorthodox to those who are not. The chapter has three parts: a discussion of the problem from the mathematical point of view; a historical survey of the development of notational systems; and a more standard review of the literature bearing on the topic.

The Mathematical Perspective

From the mathematical point of view the first several years of mathematics education are devoted to the study of the natural numbers (positive integers or whole numbers). Elementary school children study three conceptually distinct aspects of these numbers: (1) using them in counting to answer the question "how many?" (2) performing the arithmetic operations in order to short-cut one-by-one counting (addition, subtraction, multiplication, and division); and (3) learning the notational system and exploiting it to facilitate computation (Braunfeld, 1979).

The notational aspect of number is described in mathematics with two constructs: base and place value. The decimal number system from this perspective is a system of notation for real numbers[1] that uses the base 10. The base of a number system is the number of units in a given digit's place which has to be taken to denote 1 in the next higher place. In the base 10, therefore, ten units in the units place are denoted by 1 in the next higher (tens) place.[2]

Place value is implied in this definition of base. It is the property of the notational system which allows us to use a limited number of digits (ten digits in the sequence 0, 1, 2, 3, 4, 5, 6, 7, 8, 9) to record any number, no matter how large or how small, by written position. The value of the digit is determined by written position relative to the decimal point. The digit 1 written in the units position (to the immediate left of the decimal point) denotes one unit; the same digit written in the place to the left of the units denotes ten; the same digit written in the position to the left of the tens place signifies one hundred; the next position one thousand, and so forth.

This idea is mathematically expressed in terms of powers of 10. Figure 1 shows the correspondence between place (position) and powers of 10 (numerical value) for the arbitrary sequence of digits 3,125.46. Note that any number to the zero power is, by definition, equal to 1. The digits to the right of the decimal point indicate negative powers of 10. Also note that the zero has a function as a place holder, that is, it serves to distinguish 304 from 34 by "holding open" the tens place, showing an absence of any tens. The digit 3 is "held" in the notational place where it will signify 3 hundreds rather than 3 tens.

As simple as this system is, its development required forty centuries of human thought and use. It is instructive to review the path

10^3	10^2	10^1	10^0	.	10^{-1}	10^{-2}
3	1	2	5	.	4	6

3 stands for 3×10^3	or	$(3 \times 10 \times 10 \times 10)$	or	3,000.
1 stands for 1×10^2	or	$(1 \times 10 \times 10)$	or	100.
2 stands for 2×10^1	or	(2×10)	or	20.
5 stands for 5×10^0	or	(5×1)	or	5.
4 stands for 4×10^{-1}	or	$(4 \times 1/10)$	or	.4
6 stands for 6×10^{-2}	or	$(6 \times 1/100)$	or	.06
				3,125.46

Figure 1. The relationship between digits, written position, and numerical value

which mankind took in creating the modern system of notation not only for its historical interest, but also for clues it might yield regarding the difficulties which children have as they reconstruct the system for themselves.

The Historical Perspective

The Origins of Numeration

Historians of mathematics generally agree (1) that some form of counting probably served as the first mathematical element in all cultures; (2) that number words were adjectives used to describe something concrete before they became nouns signifying an abstract concept; and (3) that written notation developed in the service of keeping track of counted collections. These speculations are a direct consequence of the fact that the origins of numeration predate the oldest known artifacts that we surmise were used to keep track of counted collections.

Counting

Anthropologists have found some form of counting among the most primitive of cultures they have studied (Wilder, 1968), though some South American tribes reportedly have no unique number words, or no number words beyond one, two, and sometimes three (Conant n.d., in Newman, 1956). Conant (1896, in Wilder, 1968; in Newman, 1956) argued that oneness and twoness had a special status in earlier times and used two linguistic observations in support of that view. The first is that in some languages, the words used for the smaller ordinals has a different form from those employed for the larger ordinals. The English "first" and "second" rather than "oneth" and "twoth" (comparable to fourth, fifth, etc.) is an example. The second is that "the Indo-European words for 3 — *three*, *trois*, *drei*, *tres*, *tri*, etc., have the same root as the Latin *trans*, beyond" (1896, p. 76, in Wilder, p. 39). Everything above one and two appears, therefore, to have been designated by a single word meaning "many" or "beyond."

A different emphasis is offered by Dantzig (1967). He proposes a finger-counting theory of early number development and marshals two linguistic arguments in its favor. The first consists of instances in which the word for five is identical to, or bears a strong resemblance to, the word for hand.

> Compare the Sanskrit pantcha, five, with the related Persian pentcha, hand; the Russian "piat," five, with "piast," the outstretched hand... (In many primitive tongues) the number "five" is expressed by "hand," the number "ten" by "two hands," or sometimes by "man." Furthermore, in many primitive languages the number words up to four are identical with the names given to the four fingers (p. 10).

The second argument is that the base of many number systems is ten, and our ten fingers are the clearest model for that amount.

> In all Indo-European languages, as well as Semitic, Mongolian, and most primitive languages, the base of numeration is ten, i.e., there are independent number words up to ten, beyond which some compounding principle is used until 100 is reached. All these languages have independent words for 100 and 1,000, and some languages for even higher decimal units (p. 12).

Number Words as Adjectives

Several languages have different sets of number words for different types of objects. Thus number words were very likely adjectives or descriptors of something concrete before they became nouns embodying an abstract concept. The anthropologist Franz Boas found in the Tsimshian language of a British Columbian tribe seven distinct sets of number words: for flat objects; round objects; men; long objects; canoes; measures; and for counting when no definite object was referred to (Conant, 1896, in Wilder, p. 41). The more general counting words were a later development (Dantzig, 1967).

The vestiges of this practice can be seen in the Japanese language which employs different sets of counting words for people, other animate objects, and inanimate things, and different endings for flat, long, and compact objects. Thus the utility of using the base 10 and common counting labels for all objects, as in our modern numeration system, has not been obvious or universal in history.

Records of Counted Collections

To record "how much" or the plurality of a collection, one does not necessarily have to be able to count. Tallying, an early type of enumeration, relies on an intuition of one-to-one correspondence but does not necessitate ordered succession in the strict sense. Notches made on a stick, knots tied in a string, pebbles heaped into a pile, marks scratched on a cave wall, all of these methods of keeping track of numerosity presume the matching of the objects of one collection to the objects of the other. The symbols used for keeping track of matched collections amount to sets of strokes (or their equivalent in the case of knots or pebbles) rather than numerals, or separate notational marks designed to be read or made in ordered succession. The most advanced form of tallying is found in the abacus which is still in use in many parts of the world today.[3]

Numeration in this form is probably as old as private property. It is likely that its use was limited to such activities as simple trading, keeping track of flocks, recording time, making gross measurements of fields, etc. (Kline, 1972).

> Since primitive peoples settled down in one area, built homes, and relied upon agriculture and animal husbandry as far back as 10,000 B.C., we see how slowly the most elementary mathematics made its first steps (p. 3).

The origin of number names used for counting has been lost, and numeration in the form of tallying sheds no light on when the principle of ordered succession was intuitively apprehended. But for our purposes, several points can be made. First, counting concrete objects, and keeping track of the plurality of counted collections, probably motivated the development of individualized number names (verbal representation) and simple methods of tallying (graphic representation). We can see in young children's behavior a considerable joy in learning the counting words and delight in pointing to objects as they say them. We also have evidence from studies of children's informal arithmetic of the use of tally marks in keeping track of the plurality of counted collections (Lawler, 1979). Second, quantifying two or three elements may form a conceptual level in number which is distinct from quantifying collections containing as many as five elements. Skip-counting, or counting by twos (and later by fives and tens) may be a reflection of the solidity of these early conceptual levels, although it must be added that we encourage children to practice counting by these numerical groupings both in and out of school.

Third, if early number words were adjectives describing something concrete rather than nouns embodying an abstract concept, then the use of them in counting did not necessarily entail the construction of the unit (the n + 1 structure based on the Peano axioms). The distinction here is between "a unit" with a specific referent and "the unit" which is more general. It may be that children use "two" as an adjective signifying duality or pairs or sameness (as in the similarity of two objects such as shoes or mittens, or symmetrical anatomical features such as eyes or legs) before they understand the quantity two or even one. Fourth, and related to the last point, is the descriptive use of numerals as ordinal labels which mark position or location rather than numerical quantities (e.g., "I live at 5 Hill Street"). Here numerals or counting words name rather than quantify. Finally, tallying may be as old as counting. But many thousands of years passed from the time when we can imagine people keeping records of counted collections to the first appearance of numerals, or specialized graphic marks written in ordered succession to indicate cardinal value or the numerosity of whole collections.

The Development of Written Numeration

It has been surmized that the introduction of tallying by writing ultimately led to the development of ideographs or specialized markings for representing number. The oldest extant records of the systematic use of written numerals are found in the mathematics of the Babylonians and the Egyptians (ca. 3,000 B.C.). In this section I will describe these systems, as well as those of the Phoenicians and Greeks, Hindus, and post-Arabic Europeans, from the standpoint of changes which these subsequent civilizations introduced into mankind's history of numeration systems.

Babylonian Numeration

The Babylonian achievements in mathematics were many, but their notation is of special interest here. The Babylonians had a sexagesimal (base 60) number system, the notation for which reflected a mixed base (10 in addition to 60). It combined a feature of tallying (in the design of the numerals) with the more important feature of positional notation (the repetition of numerals used for 1 through 9 in the notation for 11 through 19, 21 through 29, and so on up to 59). The Akkadian peoples who brought Babylon to greatness appear to have acquired elements of their mathematical system from the earlier Sumerians, whom the Akkadians conquered around 2,500 B.C. (Kline, 1972; Wilder, 1968).

The Akkadians' writing implement was a reed stylus which they impressed onto soft clay tablets. The stylus had a triangular cross section which could be oriented at different angles to the clay. To make numerals they used reeds in two sizes, and with these instruments created cuneiforms, or wedge-shaped signs for representing number. Possibly due to the limits inherent in this method of writing (impressing an object into clay), the Babylonian numerals were not composed of distinctly different marks. The numerals 1 through 9 were made by impressing the larger or smaller reed into the clay the correct number of times, albeit in particular patterns. The cuneiform for ten was a single impression made at a different orientation. Twenty was conveyed by two impressions of the mark for ten, and thirty was indicated by the addition of a third impression of the same mark. Forty and fifty were denoted by impressions similar to those used for 4 (40) and 5 (50) but at an angle similar to the sign for 10. Sixty and multiples of 60 were represented by the same cuneiform as 1, and it was left to the reader to glean from the context whether 1, 60, or 3,600 was the intended meaning.

Positional notation was used in the following manner. Eleven was written with the cuneiform for 10 on the left and 1 on the right; 12 was composed of the mark for 10 on the left and 2 on the right, and so on up to 19. Twenty-one was recorded by the cuneiform for 20 on the left and 1 on the right, and so on up to 29. The pattern was repeated for 31 - 39, 41 - 49, and 51 - 59. All of the higher numbers were combinations of this positional pattern (e.g., 70 was written 60 10 but with no space left in between; 80 was signified by 60 20; 120 was denoted 60 60; and 130 by 60 60 10).[4]

Although this positional notation was in use from at least 2,000 B.C., it was not until about 300 B.C. that the Babylonians invented a mark to indicate the absence of a digit in any one position. But the sign was used only medially; they did not have a sign to indicate the absence of a digit at the right-hand end, as in our 20. Thus their numbers were ambiguous, and the exact value of the entire numeral could only be discerned from context.

The Babylonians also used positional notation to represent fractions. But these fractions were expressed as rational fractions (the quotient of two small integers) and not expressed as decimal fractions (1/5 = .2). For example the sign for 10 when intended as a fraction meant 10/60. A few fractions had special signs (1/2, 1/3 and 2/3), but these special fractions were treated as wholes and were used in the context of measuring alone.

The sexagesimal system was one of at least two systems employed by the Babylonians. There are clear indications that a decimal system was also in use.

> It is only in strictly mathematical or astronomical contexts that the sexagesimal system is consistently applied. In all other matters (dates, measures of weight, areas, etc.) use was made of mixed systems which have their exact parallel in the chaos of 60-division, 24-division, 10-division, 2-division which characterize the units of our own civilization [e.g., 24 hours of 60 minutes each]...(M)any modifications of number symbols were in use for different classes of objects, such as capacity measures, weights, areas, etc. Among these a clear decimal system has been recognized with signs for 1, 10, and 100 (Neugebauer, 1957, in Wilder, p. 45).

Besides the ambiguities mentioned above, the weakness of the Babylonian numerals was that they were cumbersome to use. The numerals were collections of wedge-shaped forms and were not represented by unique designs for each integer. Hence large numbers were indicated by complicated groups of these wedge-shaped forms.

Wilder (1968) suggests that a need for compactness motivated the development of positional notation.

> The importance of place value notation lies in its capability for expressing numbers as large as one wishes, or as small as one wishes, in terms of the same basic digits. This was important in Babylonian astronomy for the construction of tables, but in other areas, such as the marketplace, there was no comparable need (p. 50).

In sum, these characteristics of the Babylonian mathematical system emphasize two points about numeration systems: their progressive evolution in response to necessity, and their arbitrariness.

Egyptian Notation

The Egyptians had two systems of writing numbers, one that was used on monuments, and one that was practiced in daily life. The former is a hieroglyphic system and the latter is hieratic writing. In neither system was positional notation employed. In the hieroglyphic system, each marking was a picture of some object. Distinctly different hieroglyphics were used for 1, 10, 100, 1,000, 10,000 and larger units, while intermediate numbers were formed by combining these signs. The hieratic whole numbers from 1 through 10 were written with separate signs.

Fractions were unit fractions (i.e., 1/2, 1/3, 1/4, 1/5 and so forth) and were represented by an oval (hieroglyphic system) or a dot placed above the whole number to indicate that it was to be read as a fraction (hieratic writing). As was the case in the Babylonian system, a few fractions (1/2, 2/3, 1/4) were denoted by special signs (Kline, 1972).

Phoenician and Greek Numeration

Phoenician commercial activities were extensive; thus there was a clear advantage to their developing a compact numeration system. The earliest evidence of ordinal numeration is found in their system as well (Dantzig, 1967). According to Dantzig,

> The Phoenician origin of both the Hebrew and the Greek numeration is unquestionable: the Phoenician system was adopted bodily, together with the alphabet, and even the sounds of the letters were retained (p. 24).

Borrowing from the Phoenicians, the Greeks solved the problem of cumbersome notation by giving each integer a separate and distinct sign. These signs were composed of the letters of their alphabet. The older Greek numerals were not as well cipherized as the later Ionian system was (Smith and Ginsburg, n.d., in Newman, 1956). But from about 600 B.C., the Greeks used the first nine letters to represent the integers from 1 to 9. The second group of nine letters denoted the multiples of 10 (10, 20 ... 90). The remaining six letters of their alphabet, to which three archaic letters were added, stood for the first nine multiples of 100. To record multiples of 1,000 the first nine letters of the alphabet were used again, but they were preceded by a stroke. A new sign, the myriad, was introduced for 10,000. For larger numbers, the myriad was combined with the alphabetic letters.

The major weaknesses of this system were two. First, new notational marks had to be invented for larger and larger numbers. Second, the system

> The use of words with place value began at least as early as the 6th century of the Christian era. In many manuals of astronomy and mathematics, and often in other works in mentioning dates, numbers are represented by the names of certain objects or ideas. For example, zero is represented by "the void" ... or "heaven-space" ... one by "stick" ... "moon" ... "earth" ... "beginning" ... in general, by anything markedly unique; two by "the twins" ... "hands" ... "eyes" etc.; four by "oceans," five by "senses" ... six by "seasons" or "flavors"; seven by "mountain" ... and so on. These names, accommodating themselves to the verse in which scientific works were written, had the additional advantage of not admitting, as did the figures, easy alteration, since any change would tend to disturb the meter (Smith and Karpinski, 1911, p. 38).

The alphabetical systems of numerical representation were similar in that they identified numbers by letter names, and were written and read with place value. But the systems did not employ single unique letters for each number, as did the Greek system. Rather, several letters could stand for a particular number, and the specific letter adopted was chosen because it helped to make a word (mnemonic device to aid in calculating) or because the sequence fit into the rhythm of a verse.

What these systems lacked were graphic signs, so that despite the fact that zero was represented with words or letters, this did not in turn have an impact upon the writing of numerals. Nonetheless, the idea of zero was known and discussed by the seventh century.

> Brahamagupta, who lived in Ujjain, the center of Indian astronomy, in the early part of the seventh century, gives in his arithmetic a distinct treatment of the properties of zero. He does not discuss a symbol, but he shows by his treatment that in some way zero had acquired a special significance not found in the Greek or other ancient arithmetics ... [Another manuscript, ca. 830 A.D.] while it does not use the numerals with place value, has a similar discussion of the calculations with zero (Smith and Karpinski, 1911, pp. 52-3).

did not lend itself to fractional representations comparable to our decimal fractions. To record fractions, the Greeks relied on the Egyptian system of unit fractions (Wilder, 1968).

Hindu Numeration

The first evidence of Hindu numeration appears in cave inscriptions of around the third century B.C. The Hindu numerals were of many different forms, possibly reflecting the division of society by the rigid boundaries drawn between castes. Among these forms, three distinct types have been identified (Smith and Karpinski, 1911). One type was composed of simple marks, and beyond noting the simultaneous presence of this type with the other more elaborated forms, it is of little interest here.

A second type, the Brahmi numerals, are probably the forms from which our present system (that is, our "Arabic" numerals) developed (Smith and Karpinski, 1911). Fragmentary examples of early Brahmi numerals have been found in cave inscriptions in various parts of Southern India dating from the third century B.C. The Brahmi numerals were modified many times during succeeding centuries. In caves dating to the first or second century A.D., examples of the individual signs used for the numbers 1 through 10, as well as multiples of 10, 100, 1,000, and higher numbers have been located. These provide evidence for the systematic treatment of number in a decimal system. Later forms of these numerals (ca. 200 to 600 A.D.) have been found on metal coins and property deeds which were preserved because they were written on copper plates. It should be noted that the Brahmi numeration system used no zero and no place value.

A third type of Hindu numeration had place value but was composed of words and letters rather than a separate set of numerical marks.

How, when, or by whom the zero symbol was introduced into the numeral system is unknown. The elite of mathematicians may have known the zero even in 500 A.D., and they may have had some way of representing it in order to distinguish its arithmetic properties. If that were the case, it would still be possible that the merchants and common people who performed services and kept records did not grasp the significance of the novelty until much later. At any rate, the earliest known zero symbol used widely by the Hindus was a dot indicating a blank (place holder); later it was replaced by a small circle or oval. The Hindu circle resembled the Arabic notation for 5 and was therefore not immediately adopted by the Arabs. The Hindu representation did spread elsewhere, however. In China the first definite trace of zero is found in a scholarly work of 1247 A.D. "...the form is the circular one of the Hindus, and undoubtedly was brought to China by some traveler" (Smith and Karpinski, 1911, p. 56).

Following the introduction of zero into the written system, not merely as a symbol to indicate the absence of a number (place holder), but as a symbol that was treated as a number (when this occurred is a matter of speculation), it became possible to designate any quantity by the positional notation of ten digits: the notational marks for the quantities one through nine and zero. The notion that "nothing" could be a number, or subjected to treatment as if it were a number, thus took some forty centuries to evolve. As Dantzig (1967) remarks:

> ...the influence of this great discovery was by no means confined to arithmetic. By paving the way to a generalized number concept, it played just as fundamental a role in practically every branch of mathematics. In the history of culture the discovery

> of zero will always stand out as one of the greatest single achievements of the human race (p. 35).

European Developments

The diffusion of the Hindu notational system, through the Arabic world and into European cultures took place along scholarly and trade channels, but the process was gradual and met considerable cultural resistance along the way. A statute of 1299, for example, forbade the bankers of Florence to use the Arabic numerals and insisted they retain the Roman ones instead (Wilder, 1968). Dantzig (1967) notes the struggle between those who performed calculations on the abacus (an instrument which does use place value but which is not linked with place value notation) and those who used the notation.

> Today, when positional numeration has become a part of our daily life, it seems that the superiority of this method, the compactness of its notation, the ease and elegance it introduced in calculations, should have assured the rapid and sweeping acceptance of it. In reality, the transition, far from being immediate, extended over long centuries. The struggle between the Abacists, who defended the old traditions, and the Algorists, who advocated the reform, lasted from the eleventh to the fifteenth century and went through all the usual stages of obscurantism and reaction. In some places Arabic numerals were banned from official documents; in others, the art was prohibited altogether (p. 33).

As far as we know the decimal place value system for integers was not extended to fractions until the fifteenth century. The form of fractions used by the Greeks and the Babylonians in their scientific work (that is, fractions expressed as ratios of two whole numbers) persisted as the dominant method of recording fractional quantities. The first systematic discussion of decimal fractions (e.g., three-fourths expressed as 0.75 rather than 3/4) is found in the work of the Dutchman Simon Steven (La Disme, or Decimal Arithmetic, 1585). He advocated the use of decimals, as opposed to the sexagesimal system, for writing and operating with fractions, and he called for a decimal system of weights and measures as a time saving and labor saving measure for bookkeepers (Kline, 1972). Decimal fractions were not universally adopted in Europe until the eighteenth century (Wilder, 1968).

Conclusion

From this historical survey of the development of our numeration system, several points emerge which are worth bearing in mind when studying children's learning of the system.

(1) Our decimal system of number notation is one among many which have been developed to record number. Other systems have used the base 10 without place value (the Chinese notational system is a particularly clear example and will be described in Chapter III); still others have used positional notation and bases other than 10 (the Babylonian system had both a sexagesimal and decimal character).

(2) Ten is the base of many numeration systems, most likely because of the biological "accident" that people are endowed with ten fingers. Fingers have been an important tool in the history of representing number.

(3) Our system took many centuries to evolve. The systems preceding ours tended to have either a more cardinal (e.g., Egyptian hieroglyphics) or more ordinal (e.g., Greek alphabetic letters) character.

(4) Changes within any numeration system, when they occur, seem to have come about from contact between cultures rather than from indigenous

or internal developments; they seem to have resulted from one culture incorporating elements of another culture's notation system into its own, rather than from spontaneous modifications of the system from within.

(5) Often two systems were used side by side, in different specialized contexts. A contemporary example of this juxtaposition of systems may be found in measurement: the metric system, adapted from continental Europe, is widely used in scientific contexts and even taught to school children for use in that context. But cooking still uses teaspoons and cups; real estate employs square footage and acreage; and farming relies on bushels and bales.

(6) Zero was originally a mark to indicate a missing number (the medially used symbol of the Babylonians, ca. 300 B.C.); it functioned as a place holder. While the idea that zero could be treated as if it were a number, as if it signified a quantity, was known to the Hindus of the sixth or seventh century A.D., no notation corresponding to the idea has been found. Who was responsible for the introduction of the idea into the Hindu-Arabic notational system, or when this occurred, is a matter of speculation (sometime after the sixth century but before the twelfth). It is noteworthy that zero as we know and use it is a comparatively recent invention.

(7) The conventional notational system has been universally adopted because of the ease with which both everyday and highly complex calculations can be performed in it. But we need to keep in mind that it was not until about the sixteenth century that the rules for operating on integers (algorithms for the arithmetic operations) were completed, and

that it took another hundred years or so before the algorithms were devised for ratio and decimal fractions. We tend to take the algorithms for granted. Yet it is worth recalling that, as these evolved, mathematicians devised (and later discarded) other procedures that were perhaps "obvious" to them, but would be baffling to us. For instance, Dantzig (1967) gives the following example of multiplication by "doublings" used in the Middle Ages.

Modern notation	Thirteenth century notation
46	46 x 2 = 92
x13	46 x 4 = 92 x 2 = 184
138	46 x 8 = 184 x 2 = 368
46	
598	368 + 184 + 46 = 598

Dantzig says this example shows why humanity so obstinately clung to such devices as the abacus or even the tally. Perhaps the same example shows why children so obstinately cling to their fingers and concrete objects when performing calculations. Operating in a particular notational system is far removed from the concrete reality or the idea which that system was developed to represent.

Review of the Literature

This review is broadly divided into three areas: studies concerned with children's early behavior from which inferences concerning the nature and origin of numerical ideas might be made; studies focused on children's counting and graphic representations from which inferences about their understanding of number, and the cultural tools for representing number, might be drawn; and literature portraying the ways in which we have tried to influence the course of children's knowledge-building about the numeration system in our schools. I have tried to select studies that reflect a range of points of view concerning critical aspects of these topics.

Early "Number"; Sensorimotor and Counting Activities

The sensorimotor precursors of relationships such as one-to-one correspondence and serial ordering were discovered in the course of babies' spontaneous play with a carefully chosen set of materials (Moreno, Rayna, Sinclair, Stambak and Verba, 1976). Twenty-five babies between the ages of ten and twenty-four months were observed in unstructured play situations in which the following materials were made available to them; six nesting cubes (topless containers varying in size from 2 to 8 cubic centimeters); six wooden sticks (varying in length from 5 to 10 centimeters); and six clay balls (varying in diameter from 1 to 5 centimeters). The play sessions were videotaped and lasted about 15 minutes each. Three of the twenty-five subjects were observed longitudinally (six, eight, and eleven times respectively), and 47 sessions were held in all.

Moreno _et_ _al_. discerned three levels of logico-mathematical action from among the babies' activities. At the first level (10 to 12 months), three action patterns were identified. (1) The babies tended to use different actions with different objects (tapping and hitting with sticks, examining inside and outside with cubes, and pressing and biting with clay balls). (2) They repeated sequences of actions with similar objects (e.g., picking up a box, banging it, throwing it, and then repeating the same three actions with another box). (3) They engaged in a "putting into" action (putting an object into a cube, followed by putting it into their mouth). The authors suggest that this type of action can be interpreted as the infants' attempt to verify, in relation to their own body and previous knowledge, the relationships of "container and the object contained," "inside," and "smaller than."

At Level 2, two kinds of repeated actions appeared: putting one object after another into a container (that is, placing different objects into one of the two largest cubes); and individualizing objects that were similar. In the second pattern, the baby might touch one ball after another with a stick, or put the stick in one cube after another without letting go of it.

At Level 3 (16 or 18 to 24 months), three types of actions were observed. (1) Children made collections of similar objects, without the support of a container (e.g., collecting the sticks and putting them in a specific location on the floor). (2) They nested the cubes (three or four cubes at 16 months and all of the cubes by 24 months). (3) They established correspondences. Three out of five 24-month-old babies made complete one-to-one correspondences between all six of the clay balls and all six of the cubes.

These repeated actions are fascinating from the standpoint of constructing units (unitizing objects) and number. They are highly suggestive of what will emerge in older children as one-to-one correspondence and serial order, two logico-mathematical relationships that undergird the construction of number (see Chapter III).

When children produce number words and use them to count, adults generally interpret these activities as reflecting children's acquisition of number. The next group of studies are concerned with children's counting and the relationship of counting to understanding number. Each approaches the issue differently, and together these studies illustrate the range of perspectives on what number is. The first is Gelman and her associates' work on young children's counting; the second is Stake's study of older children's counting; the third is Greco's treatment of counting as an instrument for quantification; and the fourth is Steffe and his colleagues' study of counting as one route in children's construction of numerosity.

Gelman and Gallistel (1978) distinguish between abstract entities called numbers (all members of the real number system) and number as a property of concrete countable numerosities. They argue that the principles by which one abstracts number are distinct from the principles by which one reasons about number. For them, concrete number is abstracted from reality in much the same way as any physical property is abstracted from objects. Children abstract number by means of counting, and the principles used in counting form the foundation for children's number.

A variety of studies done by Gelman and her students demonstrated that children from the age of two-and-a-half or three can reliably count from two to five objects, and by the age of five can accurately count up to about ten objects. Gelman and Gallistel posited a set of five principles embedded in counting, and they used those principles to describe the progressive development of children's skills. These principles, the first three of which can be found in children's counting of small numerosities (from two to five objects), are as follows:

1. One-to-one principle. A unique number name or tag is assigned to each object in a collection;
2. Stable order principle. The tags that make up the number word sequence are applied in the same order in all instances of enumeration;
3. Cardinal principle. The last tag used is singled out to represent the numerosity of the whole;
4. Abstraction principle. The three counting principles can be applied to any collection of entities;
5. Order irrelevance principle. Any object can be used to begin the count, for the order in which the objects are counted is irrelevant.

Stake (1980) studied older children's counting in relation to many aspects of their mathematical understanding, including numerosity, partitioning and subitizing, place value, and multiplication. Using the clinical interview technique, she found that children from five to nine years of age did try to apply Gelman and Gallistel's principles, but that serious problems arose as they learned two or more sequences of number names.

One can think of counting backwards and skip-counting (counting collections by 2's, 5's, or 10's) in terms of learning additional sequences or lists of tags. Stake found that some children use the one-to-one principle (one object, one number name) in an absolute way when skip-counting: they count a collection of twenty-five objects by one's and two's, and arrive at final tags of twenty-five and fifty, respectively. When counting the same collection by five's, they might use the sequence of "five's number names" for the first twenty objects before changing to their list of one's for the last five objects (i.e., 5, 10, 15 ... 100, 101, 102, 103, 104, 105). Thus the same collection could be called twenty-five, fifty, and one hundred and five (in accordance with the cardinal principle), without any concern for the fact that the numerosities represented by these different words were not the same.

Stake suggests that Gelman and Gallistel's principles need to be expanded in order to explain the relationship between skip-counting and number. She specifies the following additions (1980, pp. 12-13):

> (1) The collection of all objects to be counted needs to remain fixed in size during the counting...and the person counting needs to have devices to partition the collection to be counted from other collections in order to prevent loss or gain.
>
> (2) The step size of the list of number names to be used (the step size is 5 when counting by fives) must equal the quantity of each subset of objects to which a single tag is assigned...
>
> (3) The final tag used to announce the cardinality of the set must be compatible with other evidence of its cardinality ...

Greco (1962) used set sizes larger than seven to study the development of counting as a tool for establishing equivalence between two sets which were not in spatial one-to-one correspondence. For him, as for Piaget, counting cannot be an instrument of quantification until the child knows how to use it as a tool for establishing equivalence between collections. With this in mind, he designed several tasks that elicited some interesting conflicts between counting and numerosity. For example, he placed two rows of seven chips in front of the child, with the second row "overhanging" the first row by one chip (that is, the spatial frontier was the same at one end of the rows but different at the other end). The child might affirm that they were the same number but insist that the longer row contained more. When presented with two rows, one of which contained eight densely packed chips and the other, seven spread-out chips, the child might say that eight is more than seven, but that the spread-apart row contained more. Finally, the child might affirm inequality in the last situation but deny the possibility of establishing equivalence by adding one: that row would be more "because you added one."

Greco found three levels in children's use of counting. At the first level (about five years of age) counting was an exercise game. At the second level (roughly age six) the child preferred to use his perception, although it did occur to six-year-olds to use counting for verification. It was not until the third level (about the age of seven) that counting had become a reliable tool for establishing equivalence.

The last approach to children's counting is centered on the construction of unit items, or successive kinds of countable items (Steffe, von Glasersfeld and Richards, 1981). In the context of a long-term learning experiment with young elementary school children, Steffe et al. studied the role of counting in the development of numerical ideas. Their epistemological stance is explicitly constructivist, and while they share many commonalities with Piagetian views, their focus of analysis is different. From their perspective counting requires not only the recitation of the standard number word sequence (SNWS) alongside the matching of one word to one item; it involves some ideas of what units or countable items are as well.

In contrast to Gelman and Gallistel, the authors argue that it is wrong to presume that units are givens in external reality. Out of an experiential background that is initially undifferentiated with respect to plurality and discreteness, or the items that make up pluralities, the child constructs a notion of what discrete (and thus countable) items are. From the point of view of the subject, that experiential background includes all sorts of sensorimotor experiences and signals by means of which he comes to distinguish among objects and events. What he can distinguish at an early age (recall the sensorimotor activities of 24-month-old and younger toddlers with respect to individualizing objects, making correspondences and ordering objects), he gives focused attention to at an older age. In order to account for the ability to perceive discreteness among objects, von Glasersfeld (1981) postulated a "pattern of attentional pulses." This model is different from, but not necessarily incompatible with, the results of the study by Moreno et al.

Steffe et al. point out that numbers are units that are composed of other units. Their construction, then, involves a second type of abstraction, and that is the creation of composite units (e.g., the cardinal number "four" or "ten") made up of other units (one, one, one...). The units or items or "ones" have to be discrete, repeatable entities that can be combined or joined together to make other composite units, or cardinal numbers. Thus counting to discover (Gelman and Gallistel), impose (Greco), or compose (Stake) numerosity, involves two levels of abstraction or conceptualization. Furthermore, in counting, the SNWS has to be coordinated with two kinds of figural patterns. The first type consists of spatial-visual patterns (subitizing, or apprehending a pattern of items arranged in space). The second involves temporal-rhythmic patterns (e.g., the kinesthetic and proprioceptive feedback that comes from putting out fingers to count, or touching items, or the auditory and motor rhythms established, for example, in the process of counting by fives or tens).

The movement from sensorimotor to conceptual units in the context of counting is described by the authors in a series of five counting types. These types, from which a parallel "ontogenesis of countable items" emerges, are based on inferences concerning the locus of children's attention in the act of counting.

1. Perceptual unit item. Children need things in their perceptual field in order to count them.
2. Figural unit item. Children can visualize or re-present an object that is not immediately perceptible; the figural representation substitutes for the perceptual item.
3. Motor unit item. Proprioceptive/kinesthetic signals, originating from physical movements, can substitute for perceptual items and thus be used as countable items.
4. Verbal unit item. The utterance of a number word that accompanied the motor act can be taken to stand for a countable perceptual item. Number words represent items to be counted.
5. Abstract (conceptual) unit item. Children become wholly freed from their dependence on sensorimotor experience. In a counting task, an uttered word is understood to imply the utterance of the number words preceding it (e.g., 4 + 3 is solved by counting on from four: "four, five, six, seven.")

The relationship of sensorimotor activities (perceptual on the one hand, and logico-mathematical on the other) and counting to the development of number that emerges from these studies is a complex one indeed. Depending upon one's epistemological stance, one's definition of number, and one's focus of concern, number is understood as early as two-and-a-half or as late as seven or more years of age. At a minimum it appears that different kinds of experiences are called upon, and that they are restructured many times over, in the child's growing awareness and understanding of number.

Early "Number"; Graphic Representation

Two studies bearing directly on the development of numerical representation were reported in the literature prior to 1980. In the first study, conducted by Sastre and Moreno (1976), three experimental situations were devised to elicit children's spontaneous as well as conventional methods of representing quantities less than ten. Sastre and Moreno were interested in whether children would make use of their knowledge of numerical notation, acquired in school, in practical contexts that suggested that they exercise the learned capacities. Fifty children between the ages of six and ten (ten children in each of five age groups from six through ten-year-olds) participated in the study as pairs, with pairs being composed of children from the same class.

In the first experimental situation, one member of the pair was sent out of the room while the Experimenter showed the other child a quantity of objects (candy). The informed child was then asked to graphically express how many candies E had put out in front of him, so that the naive child could use the message to tell how many were there (make a drawing, write something down). The children were told that if the naive child could use the informed child's marks and put out the same amount, then each of the children would receive a piece of candy.

In the second situation, the collection of candies were visible to both children, who sat across from one another at a table. However a screen prevented the children from being able to see each other's papers. In this situation the variable of speed was introduced in order to suggest efficiency, or an economy of marks, and thereby the notion of using numerals as a quick and useful method of recording numerosity. The child who produced his message faster received a piece of candy, whether or not his message was correct in terms of the numerosity represented. This second task was repeated five times.

In the third and final situation, the experimental set-up was the same as in the task just described, but the children were explicitly urged to use numerals to communicate amount.

The children's graphic representation of quantities were analyzed into four types, with a progressive evolution in the variations grouped into Type II. Sastre and Moreno's analysis is summarized below.

Type I. The children simply made drawings, without reference to the number of elements put out. For example, three six-year-olds drew a country scene (a house, trees, clouds, and mountains), two airplanes (one on the ground and the other taking off), and a fire-related picture (firetruck, hose, three firemen and a building), to represent five, five, and seven elements, respectively.

Type II. The children made schematic, symbolic representations of quantities in one of four ways that ranged from global, figural representations to tallies of various kinds.

Type IIa. Five elements were represented by a drawing of a house, two trees, one cloud, and one sun. Occasionally "five" would be represented with a drawing of a hand (five fingers) or "eight" with a drawing of an octopus (eight legs).

Type IIb. Five elements were represented as five persons, five trees, or five of some other kind of object. The number of (identical) objects that were drawn corresponded to the number of elements put in front of the child.

Type IIc. Seven elements were communicated with a direct pictorial representation of the seven candies. The qualitative properties of the elements were preserved.

Type IId. Tally marks of various kinds (squares, lines, crosses, circles, dashes) were used to represent numerosity. Graphic representations of this type were freed from any reference to the properties of the objects that were being represented.

Type III. The children wrote as many numerals as there were elements. Thus the numerals "1,2,3,4,5" were written to represent five pieces of candy. There seemed to be some necessity to identify each object with a separate mark, for some of these children rejected the suggestion that a single numeral could better represent the whole.

Type IV. The children used a single numeral to represent the total quantity of objects, e.g., "5" for five elements.

Sastre and Moreno found that in general, children's representations using Type I and Type II ideas (drawings in which numerosity was communicated by e.g., five of anything, five of the same object, an iconic representation of the five objects, and tallies) declined with age. Their utilization of Types III and IV (numerals) generally increased with age. The authors expressed surprise at finding as many children as they did who did not spontaneously use numerals to communicate quantity in the first two experimental situations. All of the children had of course been taught about them in school. Sastre and Moreno went on to wonder whether we (adult authorities) are overlooking children's intellectual functioning by not developing their capacity to use the operations on which their learned (taught) concepts are based. They suggested that we might be putting children on the road to intellectual alienation very early, by asking them to sacrifice their own reasoning for that of the adult.

The second study, conducted by Allardice (1977a, 1977b), focused on younger children's informal (unschooled) ideas concerning the representation of four mathematical ideas. The concepts included cardinality (or more specifically, representing set numerosity), relative quantity (representing one of two quantities as "more" than the other), addition and subtraction operations (incrementing or decrementing a collection of three or four objects by one or two), and temporal order (portraying the sequence in which three objects were displayed, and recording the order in which a single object was moved from one place to another in space).

This study is of interest here because it enriches the description of the development of numerical representation as given by Sastre and

Moreno. Allardice's technique for eliciting children's representation seemed to allow them to use a somewhat different set of ideas than Sastre and Moreno's situations did. Her findings with respect to young children's representations of small amounts (numerosity) and large differences (relative quantity) should therefore be considered alongside the findings of Sastre and Moreno.

Allardice's 81 subjects ranged from three-and-a-half to seven years of age and were distributed among four age groups: three-, four-, five-, and six-year-olds. She used a stuffed dog that resembled the comic strip character, Snoopy, and positioned him at the far end of the table with his back turned to the child and experimental materials. Her technique, described in greater detail below, centered on asking children to send messages to Snoopy so that he could be informed of what was taking place behind his back.

The procedure was in three parts: a screening session in which information regarding the subjects' understanding of the ideas under investigation was collected; an experimental session in which each of the four concepts was studied; and a probe session held about one week later to clarify ambiguities, to see whether giving children feedback on Snoopy's inability to understand their representations would lead them to modify their productions, and to find out whether suggesting (teaching) an alternative informal method would lead them to adopt that method. The experimental tasks are described below (taken from Allardice, 1977a, pp. 36-45).

Each of the four concepts were studied in two ways: production, which was described as children's encoding or representing of ideas; and recognition, which was also called decoding or "reading" of various materials. For the first concept, cardinality-production (set numerosity), objects varying in type (toy mice, buttons of different sizes, etc.), layout, numerosity, and size were placed on the table, and S was asked to "put something on your paper to let Snoopy know how many there are." In three of the trials the set sizes were kept small (three or four objects), and in the remaining two trials, larger amounts were used (19 chips and 11 paper clips for the two preschool groups, and 29 chips and 19 paper clips for the two older groups). The recognition tasks consisted of showing S cards, with two Xs, three triangles, or four tallies, or the numerals 1, 2, 3 or 4 written on them. S was told, "Snoopy would like you to put this many chips on his plate."

The relative quantity-production tasks asked S to make a judgment about which of two unequal quantities had more, and then to "put something on your paper to let Snoopy know that one has more." Two of the trials used large quantities of paper chips pasted on paper plates (in the ratio of 12:24 and 15:25); two involved small amounts of marbles (3:6 and 4:7); one used unequal amounts of water in identical glasses (continuous rather than discrete quantity); and one used differing amounts of gravel in glass tumblers (can be considered either continuous or discrete). The recognition portion consisted of showing S pairs of pictures (two glasses containing unequal amounts of juice, two plates on which 16 and 6 paper dots were pasted, etc.). S was asked to construct similar displays.

The operations-production portion involved three addition and three subtraction trials. For addition, E showed S a container with, e.g., two toy owls, and said, "See, here we have two owls. Now I'm going to put two more in here. You put something on your paper to show Snoopy that _first_ we had two, and _then_ I put two more." Subtraction was approached in a similar way: "See, we have three frogs (in a cup). Now I'm going to take two away. You put something on your paper to show Snoopy that _first_ we had three, and _then_ I took two away." The _recognition_ portion revolved around showing _S_ pictures from first grade math texts that were designed to show addition and subtraction operations by depicting people coming together, animals leaving a group, objects being taken away, and so forth. _S_ was asked to tell a story about the pictures.

The _temporal order-production_ tasks, which Allardice also called "ordinality tasks," were of two kinds. One involved showing _S_ three different objects, one at a time, and asking _S_ for something "to show Snoopy which one I showed you first, and which one I showed you next, and which one I showed you last." The second kind involved representing direction in space as well as order in time. The child was asked to show Snoopy the order in which a toy frog leaped across irregularly placed (paper) rocks on a two-dimensional display showing a (paper) pond with three or four rocks. The _recognition_ tasks used three objects drawn either horizontally or vertically. The child was told that it was another child's message. _S_ was asked, "Which thing do you think I showed him (her) first, and which one was next, and which one was last?"

The remainder of this discussion of Allardice's study will focus on her findings regarding the representation of set numerosity and relative quantity. The other concepts that she studied (representing addition and subtraction operations, and representing temporal order and direction) introduce the issue of recording actions or events that take place over time. While this is of great interest, it goes beyond the specific concerns of this study.

Allardice found that children employed six distinct methods in representing set numerosity, and relative quantity. Some of these methods were used by children singly, while others often appeared in combination. My interpretation of her categories follows.

Category I. Many of the younger (preschool) children made "writing-like" responses that consisted of scribbles or letters or the alphabet. Some of her subjects said that they were "writing"; some attributed meaning to their marks (e.g., "it says three" or "it says four owls"); others said that Snoopy would know what their marks meant, even if they themselves could not interpret them, because Snoopy could read.

Category II. Some of the younger children drew a single circle, and when they did, it usually took up the entire piece of paper. Children who responded in this fashion tended to make the identical mark for all of the cardinality (numerosity) trials, thus implying that they were not representing variations in numerosity.

Allardice considered Categories I and II to be "non-representational," in the sense that they did not serve to communicate numerosity to Snoopy. From the communicative standpoint, however, it is noteworthy that some rudimentary knowledge of writing-like marks as being something to be read is present, particularly in Category I. From the age of five upwards, no child made "non-representational" responses.

Categories I and II accounted for the majority of responses among three-year-olds. Recall that Sastre and Moreno's lower level subjects (who were chronologically closer to Allardice's oldest subjects) made *drawings* in which no apparent attention was given to representing numerosity (Type I), or in which the observer had to be privy to the notion that the number of elements used in the drawing was meant to communicate numerosity. In Type IIa, the elements themselves formed a picture, whereas in Type IIb, they did not (e.g., five people to represent five elements, suggesting aspects of drawing and tally ideas). These lower-level ideas are quite different in character, and they come from children of different ages. Although knowledge of the difference between writing (using alphabetic marks and numerals) and drawing is present at an early age (three years and younger), it appears that the differentiation needs to be made at different levels. The idea that marks resembling other things (Sastre and Moreno's Type IIa and IIb) can be interpreted by others in a number of different ways, and that numerals (or even tallies) are peculiarly well suited for representing numerosity *because* of their shared meaning is a later idea. Ideas concerning when (in what context) and why (for what purpose) it is more appropriate to use one or the other seem to develop over time. The instructions or experimental techniques may also have played a role in eliciting these different responses.

Allardice considered the remaining four categories as reflecting attempts at representing numerosity.

Category III. Younger as well as older children made pictures or iconic representations of the objects (Allardice included in this category children who traced the objects themselves). The younger children tended to make iconic representations for both numerosity and relative quantity, while the older subjects tended to make pictorial representations for relative quantity, and to use numerals for representing total amount. For children who cannot spell words or construct graphs to represent the notion of "more or less," the idea of making drawings seems most reasonable. Category III is similar to Sastre and Moreno's Type IIc, suggesting that this mode of representation persists as an option for communicating quantitative ideas (numerosity and relative quantity) for a good long while.

Category IV. Allardice considered circles, tallies, Xs, or other marks that were not iconic to the objects as reflecting similar ideas, just as Sastre and Moreno had done (Type IId). Among the four-year-olds, the number of Category IV responses increased for cardinality tasks while the number of Category I, II, and III responses declined.

An interesting predicament occurs when the straight-line quality of tally marks clashes with the appearance of the numeral 1. In the probe session many of the older subjects, to whom the tally method was suggested, refused to accept the tally, because "Snoopy would think it was 'one, one, one'" or would think that it was just three lines. One six-year-old protested, "You don't have any number or any owls." To my mind there is a strong suggestion that each of these children had yet to put together some important ideas with their other notions about numerical representation. The first child had yet to coordinate the inclusion relationship (one, one, one is three, when inclusion is coordinated with ordering), at least in the context of written number. The second child had yet to understand that lines or any other discrete items can be quantified by those who understand number. The third child's idea (expressing three-ness exclusively by means of pictures or numerals) may have been tied to textbook conventions in which Type IIb representations abound. Numerals (or written number names) communicate the idea of three less ambiguously, but as we have seen, they by no means exhaust the ways in which the idea can be communicated.

Category V. Older children (five- and six-year-olds) often used numerals to represent amounts, but they used them in two very different ways. The five-year-olds tended to represent each item in the display with a separate numeral (Sastre and Moreno's Type III), and six-year-olds more often used a single numeral to represent the total (Sastre and Moreno's Type IV). Twenty percent of the four-year-olds used numerals in some way. Among the five-year-olds, 76% (16 out of 21 children) used numerals, and of those children, about 70% (11 out of 16) used a numeral to represent each object (1,2,3,4 instead of the single numeral 4). Allardice comments, "The numerals were used similarly to the way other Ss used tallies; in a one-to-one relationship with the objects" (1977a, p. 67). Some of these children may also have had the idea that numerals labelled or tagged items in a display.

Amongst the five-year-olds in particular, Allardice noticed some additional comments such as "I'll write 9 because I don't know how to write 19." She also found children, who knew how to write only a few numerals, making those particular marks to show various numerosities, when they knew full well that they were not the correct marks. From these examples it appears that the notion that different numerals communicate different amounts has to be structured in its converse (a single numerosity can only be communicated with a single numeral) in order for the relationship between number and numeral to be well understood.

Category VI. Some of the older children used words, spelled out on the paper, to communicate quantity and/or to name the objects. The six-year-olds were the only children for whom the written word was an option. This oldest group used the greatest variety of techniques, and more than any of the other children, used two or more techniques in combination (e.g., numerals and pictures, or numerals and words). Nineteen out of 20 six-year-olds used numerals, and they tended to use them in the single-numeral way (Sastre and Moreno's Type IV). Interestingly enough, iconic representations persisted, not only for relative quantity but for cardinality as well (Category III). However the use of tallies or other non-pictorial marks had virtually disappeared. Allardice writes,

> The changes that occur across age reflect, in part, an increase in understanding about the nature of written representation. . . And changes across age group also reflect learning about new, formal ways to represent quantity with a concomitant decrease in the frequency of use of the informal (unschooled or symbolic) methods. However, even in the oldest group studied here there continues to be some use of informal methods (1977a, p. 68).

The relative quantity tasks were generally more difficult for all of the children than were the cardinality (numerosity) tasks. The three-year-olds used Category I and II methods more frequently than in the numerosity tasks. The four-year-olds most frequently represented relative quantity by means of pictures. From this age on, the choice of pictorial means for relative quantity generally increased while the use of tallies and numerals decreased. In Allardice's view, this shows that children can and do adapt their methods to what they understand as the demands of the task.

Allardice was interested in some further questions. One was whether young children could represent the idea that a display contained individual (discrete) objects. Roughly half (9 out of 20) of the three-year-olds' responses preserved the information about the presence of several discrete items. Seventy-five percent of the four-year-olds', and 100% of the five- and six-year-olds' representations preserved the individuality of the items.

In the screening session, Allardice assessed children's ability to establish one-to-one correspondence with a row of chips that she had made (n=3 up to n=9), and to enumerate or count chips by saying the counting numbers in their correct order, without skipping words, while touching a different chip with each word. A few of the youngest subjects could not count at all, but some three-year-olds were able to count 18 objects, and some of the four-year-olds were able to count at least 27 objects, correctly. On the basis of group data, she found that neither the ability to establish one-to-one correspondence between small sets, nor the ability to enumerate, was related to the ability to represent discrete items.

Yet another question concerned the accuracy with which children represented numerosity. Accurate representation of set numerosity increased with age, even for small quantities. Group data showed that three- and four-year-olds did not differ from one another in accuracy, but five- and six-year-olds were significantly more accurate than three- to four-year-olds. Allardice found no relationship between either one-to-one correspondence or enumeration and the ability to accurately represent a given quantity.

Finally, she found no evidence to support the idea that younger children consider it necessary to preserve such non-essential physical characteristics as layout, or the size of the objects, in representing set numerosity.

There is considerable agreement in the qualitative results of these two studies concerning the progression of ideas used by children in representing quantities. Where there are differences, they can be accounted for by relative knowledge of number and numerals (related to age) and by differences in experimental techniques: one study seemed to elicit more ideas concerned with drawing, while the other seemed to call forth ideas related to writing and written words. It will be recalled that Sastre and Moreno were interested in children's use of ideas that they had learned in school, in practical situations where those ideas would have been useful. Allardice was interested in children's unschooled ideas in situations that were similar to, as well as different from, those that would be encountered in school.

Both groups of subjects made graphic representations of quantity by iconic or pictorial means; both groups came up with non-iconic tallies; and both groups used numerals in a one-to-one match with the objects before using a single numeral to represent total quantity. One can't help but point out two observations. The first is the similarity between this progression in representation, and the progression from one-to-one correspondence to conservation in elementary number. The second is the correspondence between the four types of ideas found in both studies, and the appearance of these methods in the history of numeration systems, described earlier in this chapter.

Early "Number"; Instruction in School

A brief review summarizing research trends in the study of children's school-based understanding of the numeration system follows.

Van Engen (1947) published an early review of literature on place value and the decimal number system. He concluded that until the late 1930's, the treatment of this subject from the point of view of "meaning" was negligible. Most writers agreed that "meaning" was important in arithmetic but left unspecified what they meant by "meaning." The first half of the 1940's saw the publication of works that remedied this shortcoming by articulating two aspects of "meaningful arithmetic" from the standpoint of teaching children. First, pupils should be taught the basic principles of the decimal system, and second, they should be given concrete experiences from which to extract and internalize "meaning."

Van Engen elaborated the notion of meaning in arithmetic by linking the symbols of arithmetic to action: the meanings of the symbols in arithmetic basically represent action. He described the elements of an experience which he believed provided the essence of meaning:

> (They are) the overt acts that a child has performed, or has observed someone else perform, while working with a number of objects directly presented to the senses. Basically, they are operations that the child can actually perform with his hands or by means of some other bodily movements. These operational meanings become established by first watching someone else...perform the operation with the accompanying words which mean the operation. . . As the child's grasp of this meaning becomes firmer, he can set aside the objects and visualize the action. Finally, the visualization is not at all necessary; he has reached the stage of a mature response whereby all the bodily movements have been set aside. The child is now ready to establish meanings of a higher order based on the symbolization of these primary meanings (1947, pp. 65-6).

The application of this view to place value and the decimal system was straightforward. It encouraged the use of concrete materials in the learning of smaller numbers, and verbalization of the connection between those materials, the actions which were being performed, and the written formulation (signs, in the Piagetian sense). The implicit learning theory concerned the senses and action: bodily actions and physical manipulation of materials, sufficiently practiced, would be accompanied and eventually replaced by visualization of the actions; and visualizing would ultimately give way to conceptualization using symbols alone. Alternatively, visualizing actions, and objects, would be replaced by visualizing the symbols which had become embued with action-based meaning. Conceptualization could then take the form of operating on the symbols (signs) alone.

A contemporary formulation of this learning theory will underscore the essential continuity of this view of learning and "meaning."

> All ideas a child fully internalizes must be met first in the world of three dimensional things — the manipulative world. Later, the same ideas can be encountered and understood in sketches and pictures and diagrams — the representational world. Eventually, the child can work with these same ideas expressed in symbols — the abstract world.
>
> All learning begins as the senses bring a myriad of data to the mind. Some data are retained, and the child's memory bank grows — he is remembering experiences. Soon, a child can use these experiences to determine relationships that are not obvious at the outset — he is solving problems. Eventually, the child himself is able to pose problems and work towards solving them — he is making independent investigations (Wirtz, 1974, p. 37).

Wirtz advocates using concrete objects which the student can move around, such as bean sticks (sticks on which ten beans have been glued) and loose beans. After children have worked with such manipulatives, then place value notation can be introduced in the first instance as "recording what you've done," and subsequently as a handy device for keeping different amounts straight. He gives as an example, "42 is better than 312 because no one will be able to tell that you mean three tens and twelve ones." The appeal for thinking about regrouping, then, lies in the communication of precise amounts (the need to be understood by others), rather than the idea that twelve is composed of ten and two, that the 1 in 12 stands for ten, and ten plus thirty equals forty.

Wheeler (1971) studied second graders' performance on multi-digit subtraction and addition tests after having used different quantities of concrete materials to solve two-digit problems. The materials included the abacus, sticks that could be bundled into sets of ten, the place value chart, and the Dienes' multi-base arithmetic blocks. The children had studied two-digit addition and subtraction examples involving regrouping, but they had not previously encountered multi-digit examples. The children were grouped into three I.Q. levels and three "levels of abstraction" based on the number of materials they could successfully manipulate in solving the two-digit addition and subtraction examples.

Wheeler found that children who could use three or four of the materials in regrouping the two-digit examples scored significantly higher on the multi-digit written tests than children who were less proficient in their use of the materials. He found that this relationship was consistent across all I.Q. levels. The number of materials the children could handle in the two-digit examples, and their performances on the multi-digit written tests, were significantly correlated when age, I.Q., and competence with the basic number facts were held constant.

It is not clear whether the ability to demonstrate the idea of units and tens using a variety of objects indicates how well the student understands the notion of relative magnitudes (hundreds vs. tens) and/or groups of ten (150 = fifteen tens) and/or the regrouping algorithms (in the sense of "do the same procedure for the hundreds column as you've done for the tens and units columns." One suspects the latter among second graders.

The introduction of ideas based on the theory of sets dominated much of the discussion of children's arithmetic during the 1960's and early 1970's. A comprehensive yet succinct statement of the connection between set theory, learning theory, and place value ideas can be found in Payne and Rathmell's discussion of number and numeration (Payne and Rathmell, 1975). The authors discuss number from a set theory perspective:

> The main component of the concept of whole number is the classification of sets that are in one-to-one correspondence. If the elements of two sets match one-to-one, then the two sets have the *same* number. One set has *just as many* as the other...The *specific* number of members, however, is of use most often, and it needs a name, both oral and written (1975, p. 127).

They suggested various matching activities designed to underscore the similarity between sets of a given number and encouraged the comparing and ordering of sets of differing amounts. Oral names and written marks were explicitly recognized as being two different forms of conveying information about number.

Partitioning of sets was advocated as an early grouping activity that would be necessary for teaching the base 10 decimal numeration system.

They defined numeration as "those concepts, skills, and understandings necessary for naming and processing numbers ten or greater" (p. 137), and listed five abilities that were important for their learning: (1) grouping objects into equivalent sets and naming the number of groups; (2) a scheme for grouping more than once, that is, grouping ten units into a single group of ten, ten groups of ten into one group of a hundred, and so forth; (3) a scheme for recording groups, or a positional (notational) scheme; (4) representing numbers by oral number names, written numerals, and a base representation that directly indicates the number of groups (e.g., a ten-rod and two unit blocks for ten); and (5) translating from one representation (oral/written/object) to another (pp. 137-8). With the exception of the emphasis on grouping objects into equivalent sets and naming the number of groups (point 1 above), it is difficult to discern any great difference between this and non-set theory prescriptions for teaching. The implicit learning theory is substantially the same as those described above.

At about the same time, curricula and diagnostic tests based on task analyses, derived from Gagne's learning hierarchies, were being formulated (e.g., Smith, 1973; Resnick, Wang and Kaplan, 1973). Utilizing Gagne's method of beginning with a complex problem and working backwards, various problems in elementary mathematics were analyzed into simpler or prerequisite capabilities. Task analyses focus on the problem to be solved, rather than on the process of learning from the point of view of the individual child. Using this method, Smith (1973) constructed a written place value diagnostic test that resulted in his finding a series of inadequacies with respect to place value prerequisites among second graders who were low achievers in arithmetic. These children could (1) recognize sets; (2) group sets; (3) identify the cardinal number of a set; (4) group sets of ten; (5) group sets of ten with a remainder of ones; and (6) write numerals for sets of ten and a remainder of ones. What they had difficulty with was (1) interpreting the value of each place in a two-place numeral (e.g., circling the numeral in which the 6 stands for 6 tens, 63 or 36; deciding that 90 means ___ ones, ___ tens); (2) counting by 10s when the first number was not given (e.g., ___, 13, 23; ___ 65, 75); and (3) interpreting 10 ones as 1 ten and 1 ten as 10 ones (e.g., 9 tens = ___ ones). High achievers could do all of the above, but both high and low achievers had difficulty (1) exchanging ones for tens and tens for ones (e.g., 1 ten, 7 ones = ___ tens, 17 ones); and (2) interpreting, or naming the same number in several different ways (e.g., 2 tens, 13 ones = ___ ones; 5 tens, 18 ones = 6 tens, ___ ones). These last two items convey what others have called "renaming" and "exchanging ones for tens and tens for ones in regrouping" (e.g., Flournoy, 1967; Scrivens, 1968; Wheeler, 1971; Rathmell, 1972).

The results of diagnostic tests based on task analyses can help to specify particular areas of difficulty for children. But they do little in the way of shedding light on learning (construction) processes not directly linked to subject-matter descriptions. The presumption that subject-matter descriptions translate into psychological descriptions of behavior, an assumption that lies at the heart of learning hierarchies, is quite simply wrong, from the Piagetian point of view. In addition, learning hierarchies presume a positive transfer from one level to the next in the learning hierarchy, for lower level learnings are included in the higher level tasks (Resnick, 1976).

Resnick critiques this and other models of arithmetic instruction and suggests that the real contribution of task analyses for instruction lies in providing a psychological description of the competence sought, or what we think the child ought to know. She raises an interesting question for instruction: is it possible to teach "learning to learn" strategies?

> ...one appropriate concern for instruction is the possibility of teaching general strategies for invention and discovery — strategies that will help learners to be less dependent on the instructor's elegance in presenting particular tasks (1976, p. 76).

Resnick suggests that conscious use of self-questioning and self-monitoring might be one such strategy, and that instruction in this strategy might be possible to formulate (p. 78). However, there seems to be a rather large distance between "strategies for invention and discovery" and "self-questioning and self-monitoring." Techniques for the latter might be a bit more amenable to direct teaching than the latter, and perhaps that is what she means to incorporate into her approach.

It should be noted that regrouping does not have to be confined to instances where numbers larger than ten are involved. In the context of suggesting "mental regrouping" as an alternative to "counting on" strategies for addition, Hatano (1980) touched upon a Japanese curriculum in which children were deliberately schooled in quantities less than five before moving on to numbers greater than five. In this curriculum, it was not until children were well versed in the set partitioning of five that they moved on to larger quantities, and when they did, the larger numbers were taught as composites of the intermediary unit, five, and smaller, well known numbers. The Japanese children following this curriculum were taught to think about six as five and one, seven as five and two, and so forth. As a consequence of this type of instruction, these children were apparently less dependent on counting as a means of solving simple addition problems. In contrast to American children, these Japanese students regrouped in terms of five's and thus circumvented the need for "counting on" and "counting back," with its concomitant need to use fingers or some other mechanism to keep track of "when to stop the count." When approaching a problem such as $8 + 7 = ?$, they used one of the following strategies, neither of which involve counting:

(a) $8+7 = 8 + (2+5) = (8+2) + 5 = 10 + 5$;

(b) $8+7 =$ two fives $+ (3+2) = 10 + 5$.

In this curriculum the children worked with "tiles" or paper representations of quantities that combine features of base 10 materials (but with an intermediary representation for five — a strip of cardboard equal in length to five unit-squares) and the abacus (but with the five-strip replacing the bead as a representation of five). The modern abacus combines a base 5 and a base 10 place value system by using a single bead within each base 10 position to represent an intermediary number: 5×10^0; 5×10^1; 5×10^2; and so forth. Thus a number such as 278 would be shown on the abacus as 2 hundred-beads; 1 fifty-bead plus 2 ten beads; and 1 five-bead plus 3 one-beads.

It is possible that numerical quantities greater than five are more easily understood by pausing at five, and then deliberately constructing larger numbers out of that intermediary unit (five as a unit of units, in Steffe and his colleagues' terminology). It is also possible that stressing real understanding with small quantities before moving on to larger (greater than five) numbers gives children a chance to build confidence with their own numerical reasoning. Perhaps an intermediary representation (tiles) helps children to better understand larger numbers. Whatever the case, it should be mentioned that several authors have criticized the degree to which American children have been encouraged to rely on counting to solve addition problems (e.g., Wirtz, 1980) and have gone so far as to suggest that counting strategies or schemas get in the way of place value understanding (e.g., Stake, 1980).

Ginsburg (1977a, 1977b) studied children's learning of "codified arithmetic," including the notational system, by means of interviewing them about their ideas concerning how to write spoken numbers and how to read written numbers. From their errors and explanations, he inferred a three-stage process in writing numbers, and a three-stage process in understanding written number. With regard to writing numbers, he identified the following (1977b, p. 21):

Stage 0: The child fails to write numbers in any coherent way.

Stage 1. The child makes errors in writing small numbers, but these errors are a result of his/her following some kind of a simple rule inappropriately. For example, the child writes 305 for thirty-five.

Stage 2. The child writes small (better-known) numbers accurately but makes errors in writing large numbers. He combines a "large number chunk" with a familiar numeral. For example, the child writes 600,023 for six thousand twenty-three.

Stage 3. The child accurately writes numbers of reasonable size.

Ginsburg suggests that reading numerals may roughly parallel this sequence in writing numerals. What is noteworthy in this sequence is the repetition of an error, first made with smaller numbers and then reappearing with larger numbers. The errors may be based in part on the idea of "writing numbers like they sound," and in part on the difference between familiar and unfamiliar amounts.

With respect to understanding what has been written, Ginsburg found the following sequence of stages. He was interested specifically in the children's ability to link written number with a theory of place value (1977a, pp. 85-89; 1977b, pp. 21-27).

Stage 1. The child writes numbers, both large and small correctly, but cannot explain the rationale for doing so. He acts without being able to theorize, somewhat like a computer.

Stage 2. The child writes a given number in the correct way and volunteers that hypothetical alternatives (empirical alternatives) would be wrong. For example, for 13 the child may say that if you write 31, it would be a different number, or if you wrote a 2 beside the 1 instead of a 3, it would be a different number.

Stage 3. The child connects the writing of numbers with a theory of place value.

> Child 1: The child is asked why 13 is written the way it is. "Because it's one ten and three more." Where is the ten? "Right there" (points to the 1). And where is the three more? Points to the 3 (1977b, p. 25).
>
> Child 2: Allison had written 123; she was asked what the 3 means. "I don't know. It's the last number." What does the 2 mean? "It's on the 10's stick." Ten's stick? I don't see a stick there. Where do these sticks come from? "There's a stick in my head." Where do these sticks come from?

> Allison explained that the sticks were a kind of abacus device which her teacher used in teaching arithmetic (1976b, p. 26).

> Child 3: "Cause there's one 10, right? So you just put 1. I don't know why it's made like that. They could put 10 ones and a 3. So you see 13 is like 10 and 3, but the way we write it, it would be 103 so they just put 1 for one ten and 3 for the extra 3 that it adds on to the 10 (1977b, p. 27).

These interviews on place value, conducted with second and third graders, reflect children's awareness of different aspects of the notational system and its relation to number. Each child seems to have put ideas together in a slightly different way. Child 2 explains the 2 in 123 in terms of a well known object that she has used in learning about notation. In contrast to Child 1 and Child 3, she makes no explicit reference to quantity. Child 3 clearly articulates the confusion generated by knowing a lot, but not quite enough, about the notation: he understands that 10 + 3 = 13; that "ten ones is the same as one ten," or that 1 + 1 ... + 1 = 10; that recording both ten and three in terms of ones would yield a different number (103); and that 1 is written for "one ten." It seems that he needs an additional idea in order to know why "you just put 1 for one ten," and that is that zero is a place holder, in this case for units. The 1 in 10 _is_ ten and the 0 in 10 _is_ ones, or more precisely, an absence of ones. One suspects that Child 1 does not yet have this idea, either.

The relationship between number and numerical representation is far from straightforward from either the standpoint of historical evolution or from the perspective of children's development. In the next chapter I will elaborate the theoretical perspective from which this study is undertaken.

CHAPTER III

THEORETICAL FRAMEWORK AND HYPOTHESES

Adults accustomed to using any notational system have already constructed their own understanding of the principles and peculiarities of that system. But child-learners poised outside a notational system have yet to construct those linkages; they only gradually come to understand how the "squiggles" work, and the meanings they hold for proficient users. We are concerned with the process by which child-learners build their knowledge of numerical notation by putting into relation their individually constructed concepts regarding number with their knowledge of the cultural objects for representing those concepts.

The graphic marks are organized into a system. How do children construct their knowledge of the principles underlying that system? These principles are, on the one hand, numerical (knowledge of number) and on the other hand notational (knowledge of the cultural object). While the primary focus of this study is the digits and numerals of the written system, we cannot ignore the impact of the verbal or number-word system in this knowledge-building process.

Piaget never directly addressed the question of how children construct their theories about notational systems. But he did distinguish among three aspects involved in the child's representation of quantities. They are (a) the idea of number and other kinds of quantities, (b) the representation of ideas with symbols, and (c) representation of ideas with conventional signs (Piaget, 1941; Piaget and Inhelder, 1966). The first part of this chapter deals with these and related theoretical distinctions. The second part explicates the hypotheses postulated to study the problems that children encounter in the process of reconstructing, or coming to know, the numeration system.

The Construction of Number

Number resides nowhere in reality, but only in the minds of people who have created it. Although objects are observable, their number is an idea about them (Judd, 1927; Inhelder, Sinclair and Bovet, 1974; C. Kamii and DeVries, 1976; Steffe, Richards and von Glasersfeld, 1979). Number, in the Piagetian view, is an idea constructed by each individual by synthesizing two kinds of relationships: order and hierarchical inclusion (Figure 1). To quantify a collection of objects numerically, the subject must put them into a relationship of order to ensure that each will be counted once and only once (the dotted lines in Figure 2). He must also put them into a relationship of hierarchical inclusion (the solid lines in Figure 2), such that one is included in two, two in three, and so forth. As the child puts the objects into these relationships, he transforms them into "one," i.e., objects whose qualitative characteristics (e.g., shape, size, and color) are ignored and whose only salient feature is "one" or "an element." This transformation of discrete objects into "ones" undergirds the construction of the unit.

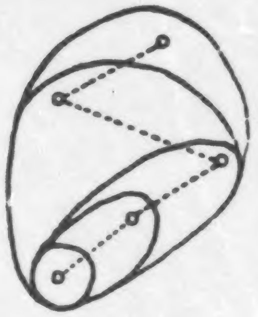


Figure 2. Relationships of order (dotted lines) and hierarchical inclusion (solid lines) involved in number.

For children who have not yet coordinated ordering and inclusion relationships, number cannot be said to exist. The absence of order can be seen in the 4-year-old who "counts" the five objects as shown in Figure 3: the child counts seven objects by skipping one and counting some of them more than once. Only when the child comes to feel the logical necessity of not skipping any, and of not counting any of them more than once, will he feel the need to order the objects mentally so he can keep track of the ones he has already counted, and those he has yet to count.

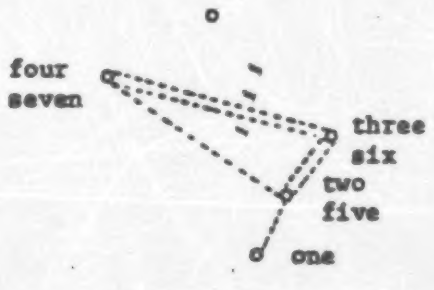


Figure 3. The absence of order in counting

The absence of inclusion can be seen in the 4-year-old who counts five objects, announces that there are "five," but points only to the last one when asked to "show five" (Figure 4). This child points only to the fifth object because for her, "one" is only a name for the first object in the series, "two" for the second object, and so forth. Counting in the manner shown in Figure 4 is much like naming individuals in a group: "Jesse, Jason, Sarah, Amy, Andrew." As the child names each person, she (for good reason) does not mentally include Jesse in Jason, etc. In the same way, she does not mentally include one in two, two in three, in "counting" the objects in a collection.

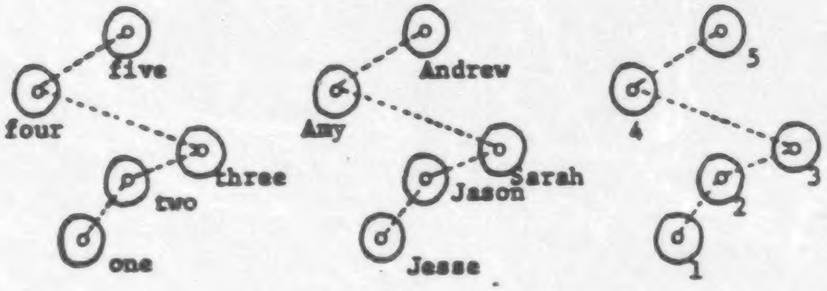


Figure 4. The absence of inclusion in "counting," naming and writing

The same problem can be observed in the child who writes the numerals corresponding to the number words but points only to the last object when asked to show what a numeral stands for ("5" in Figure 4). Here the child treats numerals as squiggles that mark individual items; they are implicitly limited to being "ordinal-labels" that show the position or location or identity of individual objects in space, but they do not at the same time show the whole (cardinal value).

Empirical and Reflective Abstraction

To explain how relationships such as order, hierarchical inclusion, and number are constructed, Piaget distinguished between two forms of abstraction: empirical abstraction and reflective abstraction. Empirical abstraction refers to the subject's deliberate noticing of specific physical properties of objects to the exclusion of other observable properties. In the well-known class inclusion task, for example, the child attends to color while ignoring the wooden-ness of the beads (Inhelder and Piaget, 1959). The observable property that the child abstracts is in the object.

In reflective abstraction, by contrast, the child creates relationships among objects, relationships such as "similar or different," and "three or third." Relationships are mental constructions that cannot be known simply by the observation of objects or events in external reality. For example, the similarity or difference between any two objects or any sets of objects exists neither in one nor in the other. These relationships are created in the mind of the person who considers the objects or sets to be similar or different. The number three is not an empirical property of objects or sets and as such cannot be abstracted from them by empirical abstraction. The designation of first, second, and third among objects or events is likewise in the spatio-temporal framework which the individual introduces into an arrangement of objects in space, or succession of events in time. Numerical relationships, once constructed, are introduced into collections of objects.

The child does not construct number as an isolated idea. Rather, she builds the idea in conjunction with other aspects of knowledge, physical and social. The indissociability of the construction of these many aspects of knowledge in the psychological reality of the child will emerge in the discussion which follows.

Physical, Logico-Mathematical, and Social (Conventional) Knowledge

Piaget made a fundamental distinction between physical knowledge and logico-mathematical knowledge. Physical knowledge is knowledge of objects in external reality, such as the fact that a blue wooden block is made of wood, painted blue, and has a certain weight. These are examples of properties of objects that are observable and that can be known in part by empirical abstraction.

Logico-mathematical knowledge, by contrast, consists of relationships that are created and coordinated by each individual. Class inclusion (the logic of quantifying by means of "all," "some," and "none") rests on the ability to determine and consistently apply a criterion of membership to a collection of objects, and is an example of logico-mathematical knowledge. A blue block and an otherwise identical red one are in the relationship of "different" when the individual focuses on their color and thus judges them to be different. The blocks are "the same" when the individual considers their other properties (shape, substance, weight, size). If the child cannot coordinate the part-whole relationship of six blue blocks and two red ones that are different in a way but the same in another way, he will say that there are more blue blocks than blocks, meaning that there are more blue blocks than red ones. When his thought has become mobile enough to be reversible, he will simultaneously put the whole (blocks) into relationship with the parts (blue ones and red ones) and judge that there are more blocks than blue ones. The child has then become able to differentiate and coordinate the parts and the whole.

The class inclusion task illustrates the indissociability of empirical and reflective abstraction, and of physical and logico-mathematical knowledge, in the psychological reality of the child. To note that a block is a block, the child presumably thinks of "blocks" in relation to other objects that he knows (physical knowledge and reflective abstraction). Similarly to note that a block is red, he implicitly thinks of red in opposition to whatever other colors he knows (empirical and reflective abstraction). The child would not be able to create the relationship of "different" (logico-mathematical knowledge) in the absence of objects that have dissimilar properties (physical knowledge). He would likewise be unable to put the blocks into the numerical relationship of "eight" (logico-mathematical knowledge in the quantification of discrete quantities) if blocks behaved like drops of water that easily join to make a single larger puddle (physical knowledge of continuous quantities).

In addition to physical and logico-mathematical knowledge, Piaget delineated a third type of knowledge, and that is social (conventional) knowledge. While the ultimate source of physical knowledge can be said to be in objects, and the ultimate source of logico-mathematical knowledge to be in each child, the source of social knowledge is in the shared conventions that have been created by people. Spoken and written language, knowledge of historical events, and familiarity with commercial weights and measures, are all examples of social knowledge. So is an awareness of how people partition a year: into calendar months; "The Four Seasons" (Vivaldi); planting and harvesting times (agriculture); football and baseball season (sports); the Sabbaths and holy days (religion), etc. Part-whole relationships are again involved in the child's construction of social knowledge. The year can be thought about in terms of different kinds of seasons, and the seasons in turn can be broken down in all sorts of ways.

The hierarchical relationship involved in the construction of number can be seen to develop in the child's daily adaptation to the physical and social world. The child learns about the properties of objects as he finds out which objects can be eaten (and which ones cannot), which edible objects taste good (and which ones do not), which objects are alike (a chair is like a sofa even though their shapes differ) and which objects are different (an orange is not like a ball in spite of their shape). The child constructs the structure of hierarchical inclusion as he thinks about the qualities of physical and social objects that are relevant to him. In the hierarchical structure involved in number, he ignores all of these qualities and puts only one object in each class.

The differentiation of knowledge into social, physical, and logico-mathematical kinds helps to clarify the different aspects involved in children's knowledge-building of the notational system. The spoken words "one, two, three" and the number-squiggle "1,2,3" are examples of social knowledge. The underlying idea of number, however, is logico-mathematical knowledge which is universal. This universality is due to the fact that logico-mathematical knowledge has its source in each individual who has created numerical relationships among objects. The quantity five is the same, regardless of what it is called and how it is written, and this quantity can be constructed with pebbles and seashells as readily as with Unifix cubes and other "math aids." The distinction Piaget made between reflective and empirical abstraction also helps to explain why children cannot learn the idea of five, merely by being presented with five cookies or five pennies or a picture of five ladybugs in an arithmetic workbook.

Logico-Arithmetic and Infralogical Operations

When Piaget focused more specifically on logico-mathematical knowledge, he made a further distinction between logico-arithmetical (aspatial and atemporal) and infralogical (spatio-temporal) operations. Let us leave aside for the moment the term, operations. Piaget conceptualized the totality of our knowledge as being organized through two frameworks, a logical framework and a spatio-temporal (infralogical) framework (Piaget, 1946; 1971). When we hear the word "Piaget," for example, we understand it through a framework that enables us to think of such categories as scholar, scientist, and psychologist. In contrast to that word, the words "Queen Victoria" bring to mind such classes as "ruler" and "monarch." These social categories (status, profession), are examples of relationships among discrete objects that are independent of space and time. To locate Piaget or Queen Victoria in the totality of our knowledge, we need a framework of space and time in addition to a logical (classificatory) framework, to know that he was not a scholar who lived in Renaissance Italy, and that she was not a medieval Chinese monarch.

Both the logical and infralogical frameworks are constructed by reflective abstraction. The child's construction of logical relationships by reflective (and empirical) abstraction has already been discussed. I will add here some examples of how children build spatial and temporal relationships by reflective (and empirical) abstraction.

Objects exist in space and time. We therefore know space and time intuitively as if they were observable physical knowledge. During the sensorimotor period, a baby learns to pull a pillow to make an object perched on top of the pillow come closer to her. Observation and experience are necessary for this learning to occur. But even this knowledge is a relationship that the baby must construct between the pillow and the object that is on top of it (Piaget, 1936). Prior to constructing this spatial relationship, it does not occur to the infant to bring the object closer by pulling the pillow. Time, too, is involved in this maneuver. Anticipatory and intentional behaviors, in fact, attest to the infant's creation of temporal relationships; to anticipate the breast when a particular person appears (empirical generalization), the baby must put observable bits of knowledge into temporal relationships.

The spatio-temporal framework that Piaget postulated is an extension of the small spatio-temporal relationships that are constructed during the sensorimotor period. Adults also create spatial frameworks in their daily adaptation to the world. Upon entering an unfamiliar building, for example, we begin creating a system of spatial relationships concerning the building as we locate the elevator, the stairway, the rest room, etc., in relation to the entrance. In addition, we try to put the entire building into relationship with the streets outside that serve as an external spatial framework. The north-south and east-west coordinates involved in the larger framework are, in turn, related to the apparent movement of the sun. According to Piaget, we create a spatial framework by reflective abstraction and later locate objects and events within this framework.

An example of the temporal framework can be found in *The Child's Conception of Time* (Piaget, 1946). A 4-year-old was asked about the relative ages of some familiar people.

> Is Erica, [your younger sister] a baby? *No, she can walk*. Who is the older of you two? *Me*. Why? *Because I'm the bigger one*. Who will be older when she starts going to school? *Don't know* ... Is your mother older than you? *Yes*. Is your Granny older than your mother? *No*. Are they the same age? *I think so*. Isn't she older than your mother? *Oh no*. Does your Granny grow older every year? *She stays the same*. And your mother? *She stays the same as well*. And you? *No, I get older*. And your little sister? *Yes!* (categorically) ... Who was born first, Erica or you? *Don't know*. Is there a way of finding out? *No*. Who is younger, Erica or you? *Erica*. So which one was born first? *Don't know*... (p. 221).

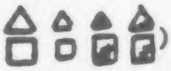
For this child, time was an intuitive notion that was known through observable facts such as people's size. Later, after creating a system or framework of time, the child will be able to locate her birth and her sister's birth within this framework, and to deduce that forever thereafter, the difference in their ages would remain the same.

Let us now return to Piaget's use of the term operation. "Operations" can be understood roughly as "reasoning logically." In Piaget's words, operations are "internalized actions that are grouped into coherent, reversible systems" (Piaget and Inhelder, 1966, p. 93). For Piaget thought is interiorized action. During the sensorimotor period, the action is indissociably physical and mental; later, as the physical aspect becomes less of a necessary support, the action can be carried out more purely on the mental plane. On the mental plane, it becomes possible to do and undo actions without dependence on physical action, and thus actions can be mentally "reversible." When the child reasons that there are more blocks than blue blocks, he is mentally cutting the whole (set of blocks) into parts (blue and red groups) and then reversing the thought to reunite the parts into a whole. When the child becomes able to perform such mental actions generally and consistently in a variety of cases, such as deducing that there are more trucks than firetrucks, more dolls than baby dolls, etc., he is said to have grouped these actions into "a coherent, reversible system." In specific cases when the child applies such coherent and reversible actions, he is said to be performing an "operation." In a word, he no longer reasons illogically as he had done when he said there are more blue blocks than blocks.

In Piaget's theory, logical operations are distinct and different from infralogical operations because the former concern relationships among discrete objects where space and time are irrelevant: there are more trucks than firetrucks, regardless of how these objects are arranged in space. Number grows out of logical operations, and arithmetical reasoning such as "6 + 2 = 8" is a refinement of the thinking involved in making such judgments as "there are more B than A." The conservation of elementary number task sets up an opportunity for the child to think about spatial (length or density of a row of objects) and numerical ideas (number of objects in the row) at the same time.

Initially two rows of objects are placed in front of the child in spatial and numerical one-to-one correspondence. Then the spaces between the objects in one of the two rows are changed so that the row is transformed into a longer (then a shorter) row. The child is asked whether either the untouched or transformed row has "more or less or the same amount to eat." Nonconservers fail to conserve number (that is, fail to maintain the idea of numerical equality when one of the two rows is spatially transformed) in part because they base their judgment on spatial frontiers. They also fail to conserve because their thought is not reversible. Spreading the objects apart and putting them back into one-to-one correspondence are two opposite material actions which cannot take place instantaneously in time. When the child's thought has become reversible, he becomes able to coordinate these ideas in his head and deduce that the number remains the same.

Infralogical operations refer to spatial and temporal operations such as those involved in measurement (the quantification of continuous dimensions). A continuous dimension such as length, for example, becomes susceptible to numerical quantification when some unit of comparison is chosen, and when that unit is repeatedly applied to the spatial dimension. The whole is partitioned into units, and each unit is added to the ones already counted. To measure a piece of paper, the child might use a ruler to cut the length or width into inches; but she also puts the inches back together into a whole that includes one inch in two, two inches in three, and so forth. The measurement of time is likewise an infralogical operation, except that time has to be represented in the individual's mind. In measuring an interval of time, the child has to mentally include one minute in two, one day in two, one year in two, etc. Measurement is thus an operation that involves reversibility and the construction of coherent systems.

As the terms "logico-arithmetical" and "logico-algebraic" imply (Piaget, 1971), number grows out of logical operations which are distinct and different from infralogical operations. The roots of logical and infralogical operations, however, are still undifferentiated in young children. This is why children at the stage of graphic collections use spatial configurations and make houses () when asked to "put together the things (geometrical shapes of different sizes and colors) that are alike" (Inhelder and Piaget, 1959). This initial lack of differentiation between pre-logical and spatial considerations was explored in the context of studying why children conserve elementary number before conserving continuous quantities (Inhelder, Blanchet, A. Sinclair and Piaget, 1975). Modifications of the conservation of elementary number task enabled Inhelder and her colleagues to infer that young children surround objects with a "spatial envelope" which interferes with their judgments about numerosity. The gradual differentiation between discrete (number) and continuous (space) dimensions shows that the roots of logical and infralogical operations are distinct but undifferentiated at the preoperatory level, and that the mechanisms ensuring the two conservations become isomorphic at the operatory level.

One could argue that spatio-temporal considerations interfere with the development of logico-arithmetic operations. But they may also help children to organize their knowledge by providing occasions for them to think logically and numerically. When we tell children to take turns, for example, we impose a social rule that utilizes the idea of temporal sequence. This serves as an occasion for children to make such logical relationships as "those who have had a turn (and those who have not)," and numerical relationships as "three more people and it'll be my turn again." In the psychological reality of the child, number develops indissociably and in interaction with spatio-temporal relationships as well as with a knowledge of objects and people.

The Representation of Number

The idea of representation has been used in many different ways in psychological research. As the term is used here, representation refers primarily to two-dimensional graphic forms. They have been differentiated by Piaget and others into two types: symbols and signs. Before discussing the representation of numerical quantities, let us consider the distinction between symbols and signs in more general terms.

Symbols generally represent reality while conserving some resemblance to the reality. Arbitrary signs, in contrast, bear little similarity to the reality being represented. A drawing of a house is a symbolic representation; the spoken word "house," and its representation with alphabetic letters, are both signs. Signs are collective signifiers; their shapes (graphic, auditory, etc.) and their conventional meanings are socially transmitted. Musical notation provides another example of a shared sign system. In Western music, pitches are communicated by marks made on a staff; duration (how long a pitch is to be sustained) is noted by modifications of the marks; dynamics, rests, accents, and other features are all recorded with distinct notational marks. The distinction between symbols and signs thus rests upon (1) the degree to which the graphic representation resembles the reality or idea being signified, (2) the source of creation or validation regarding meaning, and (3) the extent to which the notational elements form a system.

In Piaget's theory, symbolic behavior reflects the subject's tendency to assimilate reality to himself, to modify reality to make it accessible to him (Piaget and Inhelder, 1966). Symbolic representations spring from within: their source is the individual's mental activity, supported by elements of the outside world. Once the child knows a house, for example, she will manage to draw houses without being taught how to make them. In contrast, both the elements of conventional notation (number-squiggles, alphabetic letter-squiggles, musical-squiggles) and the system into which these shapes are organized, must in large measure be learned. In Piaget's view, learning of this kind takes place through imitation. Imitation is accommodation to external models: it is the tendency of the subject to modify her behavior in order to reproduce something given outside herself (Piaget and Inhelder, 1966).

In practice, some symbols are collective signifiers or shared representations; they function in the same way as signs. Highway markers warning drivers against dangerous curves convey their message in symbolic form. The markings resemble the idea being communicated, and the meaning of the markings is known to all drivers. Some elements of sign systems also resemble the idea being signified. The Roman numerals I, II, III, and the Chinese ideographs 一, 二, 三 are elements of sign systems that suggest the ideas of one, two, three. In both the Roman and the Chinese systems, it should be added, that resemblance is limited to the first three graphic marks of the respective systems.

Representation of Numerical Quantities with Symbols

Tally marks and drawings that communicate numerosity are examples of the representation of numerical quantities with symbols. When children symbolize numerical quantities, they make their representations resemble the idea (e.g., of "eight") either by making direct copies of the objects (the collection in external reality), or by inventing some collection of marks that can be "read" as eight things. The important point to note is that the child does not need to be taught how to represent numerical quantities in this way because he has invented a means for making his message apparent; he has created the marks that will stand for each object. There are neither conventional marks nor systemic properties to learn.

Sastre and Moreno (1976) studied the development of children's graphic representation of numerical quantities by asking children to work in pairs. One child was asked to work with the Experimenter (E), while the other stood outside of the room. E asked the first child to "leave a message" for the child outside of the room "so that he (the naive child) would be able to tell how many pieces of candy are here on the table — so that he will be able to use your note to put out just as many." Sastre and Moreno found that young subjects symbolize eight pieces of candy with a drawing of some other object (e.g., an octopus with eight legs) before representing that quantity either with a direct drawing of the pieces, or with tally marks to indicate the correct number. The idea of representing numerosity with numerals (the digits 1 through 8, or simply 8) didn't spontaneously occur to their subjects until a surprisingly late age.

Representation of Numerical Quantities With Signs

Digits and numerals, as well as alphabetic letters and words, can be thought of as socially constituted objects of the external world (Ferreiro, 1979). They are objects which can be thought about, and many children do have ideas about them well before they enter school (Ferreiro and Teberosky, 1979).

In the school environment numerals are discussed almost exclusively as tools for numerical quantification. But children's own ideas about the meanings of these graphic marks are often far removed from the conventional meanings. These personally constructed ideas are not aberrant in and of themselves; they become strange or erroneous only in contrast with conventional meanings. An anecdote will serve to make the point. Mark (3:6) knows the letters of the alphabet and counts with proficiency up to twelve or thirteen. He was sitting at the kitchen table, and in front of him was a jar of applesauce whose label consisted of a picture of two apples with the word A P P L E S A U C E written below. Mark knew the contents of the jar.

Interviewer: "Is 'applesauce' written somewhere?"
Mark: "Yes."
I: "Where does it say 'apple?'"
M: Pointed to the two pictured apples.
I: "Where does it say 'sauce?'"
M: Pointed to the whole word, APPLESAUCE. Then he spontaneously began counting the letters, "One, two...ten."
I: "Ten what?"
M: (Surprised) "Ten numbers."
I: "How so? Show me again."
M: Counted the letters again and said, "You see, ten."
I: "Yes, you counted that very well. Ten what?"
M: (Exasperated) "NUMBERS! ! !"

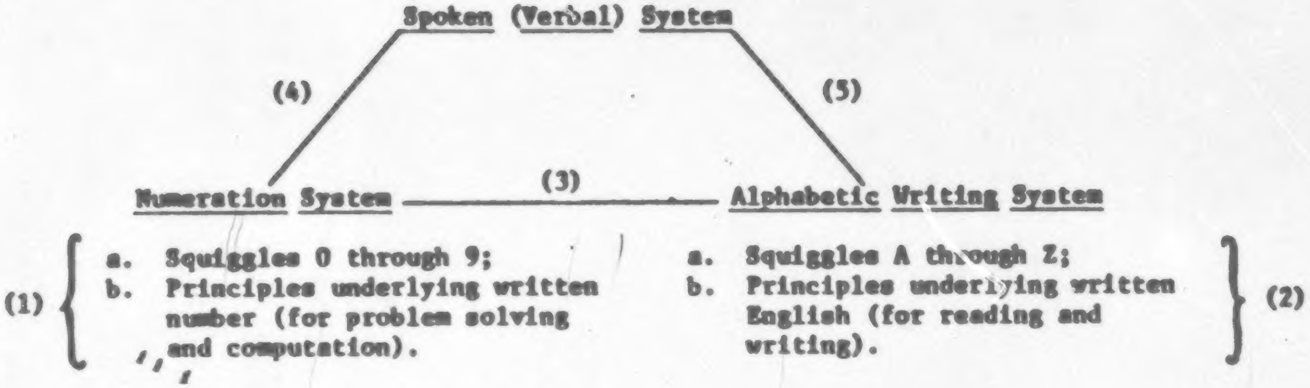
For Mark the clash between his action of counting, the identification of the things he has counted, and number is not apparent. The ambiguities will emerge as he becomes aware of the restricted meanings of letters, numerals, counting words, and number, and thus of the difference between them and his own notions.

F. Siegrist and A. Sinclair (research in progress) are studying the meanings which preschool children attribute to number-squiggles. They interviewed one child (4:1) who, when shown pictures of objects with numerals on them, thought that a 4 on a birthday cake meant "cake" and the numeral 15 on a door meant "door," as if the squiggles were the names of the respective objects. Another child of about the same age, (4:2) thought that the 4 on the cake meant "it's good to eat" and 15 on the door said "to recognize my house." For the latter child, the squiggles carried some kind of a functional message. Siegrist and A. Sinclair find that children think the numerals are linked with the objects on which they are found, but that the types of linkages, or meanings, take very different forms: *naming labels* which are redundant in the sense that they give no new information; and *functional messages* which serve such purposes as telling what the object is good for, or giving instructions or orders, in much the same way as highway speed-limit signs convey to adults how fast they are allowed to drive. In the case of functional messages, the meaning of the number-squiggles depends in part on the nature of the object on which it appears, or the frame of reference to which the object is assimilated. The latter phenomenon appears in adult functioning all the time: who would mistake "1 2 . 9 5" — the price of a book, with "1 9 8 1" — its publication date, or with that long string of numbers -- its Library of Congress designation, even though all of them are written with the same set of digits? As adults we have learned to recognize that certain socially determined constellations of digits have different kinds of meaning.

In principle, three groups of conflict emerge in the school-aged child's thinking about written number. The first group concerns clashes between spoken and written systems of representation; the second centers on conflicts between the principles underlying number and the principles governing written number; and the third focuses on differences between the ideas underlying arithmetic operations and mechanisms for written computations.

Conflicts Between Spoken and Written Systems

I have argued that both the alphabetic and the numeration systems are objects of knowledge. In this perspective, it is the task of the child to figure out not only the principles which underlie each of the systems, that is, their internal consistencies and inconsistencies, but also the similarities and differences between the systems, and the correspondences (and exceptions) between spoken and written number on the one hand, and spoken and written English on the other. These relations are summarized in Figure 5 (p.87). To record spoken language, most Indo-European languages use an alphabetic writing system. Among these languages the translation of spoken into written language rests on the phoneme-grapheme or sound-squiggle correspondence. This principle, that the graphic marks reproduce some (but not all) of the sounds of the spoken word, is alien to written number.


Spoken (Verbal) System
(4)
(5)
Numeration System
(3)
Alphabetic Writing System
(1)
a. Squiggles 0 through 9;
b. Principles underlying written number (for problem solving and computation).
a. Squiggles A through Z;
b. Principles underlying written English (for reading and writing).
(2)

(1) Numeration system (written number);
(2) Alphabetic system (written English);
(3) Similarities and differences between written systems (numeration, alphabetic);
(4) Correspondences (and exceptions) between spoken and written number;
(5) Correspondences (and exceptions) between spoken and written English.

Figure 5. Relations within and among notational and spoken (English) systems.

Most children are dreadfully confused about how to write a number such as 14 or 16. With the miserable "teen" numbers, the sound/squiggle correspondence is completely inverted: sixteen is written 1-6 from left-to-right, and not 6-1, even though the six is heard first. For the child who is also learning the alphabetic sound/squiggle correspondence in order to read, the notation for the "teen" counting words has to be mysterious. Another instance of the clash between the spoken and written systems can be seen when children write a number such as one hundred and twenty "1 0 0 2 0" — first one hundred, and then twenty, as if one were "spelling" numbers.

English uses a decimal system for writing numbers but a non-decimal system for counting words under twenty. Thirteen different words are employed for the sequence zero-through-twelve. The next seven "teen" words use either the ordinal or cardinal number names for 3 through 9 with the root "teen" affixed to them. A new word is introduced for 20, and only then does the verbal system become more systematic. The same ordinal or cardinal form used for 13 through 19, but with the root "ty" attached to them, is uttered for the decadal numbers. Twenty-one through 29, 31 through 39, etc., combine these words with the cardinal names for 1 through 9. Then new words are introduced for 100, 1,000, and larger numbers.

From the standpoint of the correspondence between the way in which numerals are written, and the number names by which they are read, the Japanese system forms an interesting contrast to the Anglo-American system. The Japanese use the Chinese system of number notation wherein 11, 12 ... 19 are written 10-1, 10-2, etc. In English there are different words for these numerals (eleven, twelve, and so forth). In Japanese these numerals are read "ju-ichi, ju-ni ... ju-ku" where "ju" is the word for 10 and "ichi, ni ... ku" are the words for 1, 2 ... 9. So instead of the new word "eleven," "ten-one" is said, and in place of "twelve," "ten-two" is uttered, and so forth. The decadal names(multiples of 10) are "ni-ju" (twenty), "san-ju" (thirty), and so forth up to "kyu-ju" (ninety), literally "two-tens, three tens," etc. The correspondence between the written and the verbal is therefore much closer in the Japanese system than it is in English.

Children are introduced to both the numeration system and the alphabetic writing system at about the same point (roughly ages four, five and six), and they often act as if the verbal, alphabetic, and numerical systems were more mutually consistent than they are. The tendency of the child who is trying to learn the rules of cultural systems is to presume that the principles underlying each of these different systems — verbal/numeration, verbal/alphabetic, and numeration/alphabetic — are more coherent than they are.

Conflicts Between Number and Written Number

The second group of conflicts which the child has to deal with concern the principles underlying number, and the principles underlying the notational system for recording number. A knowledge of the numeration system is quite separate from a knowledge of number. A child of five or six who has just recently conserved number has finally come to the important realization that the numerical quantity of a collection of objects (its numerosity, its cardinal value, its number) is completely unaffected by either the spatial arrangement of the objects, or the temporal order in which the objects are counted. So long as the child imposes some kind of order on the objects and counts each object once and only once, the number remains the same.

But the notational system employs principles that sharply clash with these fundamental properties of number. The spatial arrangement of the digits, and the left-right order in which multi-digit numerals are written, does make a difference in the quantities they represent (see Figure 6).

Spatial Arrangement	Order
1 2 5 "one" "two" "five"	215 and 251
12 5 "twelve" "five"	521 and 512
1 25 "one" "twenty-five"	152 and 125
125 "one hundred twenty-five"	

Figure 6. The spatial arrangement of the digits, as well as the order in which the digits are written, affects the numerical quantities they convey.

For the child who has only recently conserved number, and thus realized that the order and the spacing of the objects does not matter, being then told that the order and spacing of the written numbers representing those objects does matter, must be a bit bewildering.

The notation of units (squiggles and strictly notational principles) should be distinguished from the idea of base (numerical principle). Our commodious system fuses the two together, but they are distinct in spite of their correspondence. While our positional system seems perfectly reasonable to adults who are accustomed to it, it is not obvious to children nor even standard among cultures. The Chinese, for example, use a base ten system which does not depend on positional notation. This system is illustrated in Figure 7.

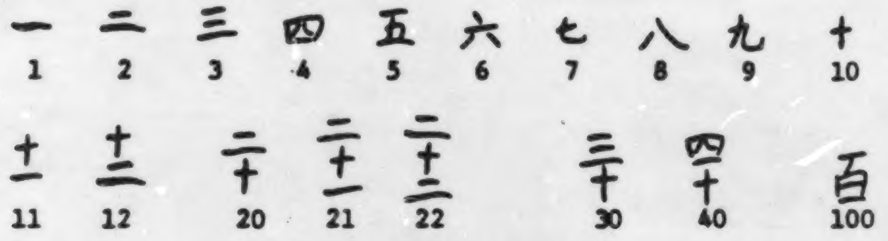


Figure 7. The numerical idea of base is distinct from the notational idea of position

Note first that "11" is easier to interpret in the Chinese system than it is in ours because the squiggle for 1 is different from that for 10. In our notational system, the 1 appearing the tens place signifies ten. Second, in the Chinese system the numerals for 11 through 19 are written so that the 10 remains visible: 11 is written 10-1 (ten and one); 12 is written 10-2 (ten and two); and so forth. Third, in the Chinese system the decadal numbers (multiples of 10) are written so that the 10s remain visible. The system which is "additive" for 11 through 19 becomes "multiplicative" for decadal numbers up to 90 (iterate ten two times, iterate ten three times, etc.). For 21-29, 31-39 ... 91-99, the two systems combine: 21 is written 2-10-1 and 22 is recorded 2-10-2; iterate ten two times and add one, iterate ten two times and add two.

The Chinese system introduces a new sign for one hundred, whereas the positional system introduces a third column (hundreds place).

Conflicts Between Arithmetic Operations and Written Computations

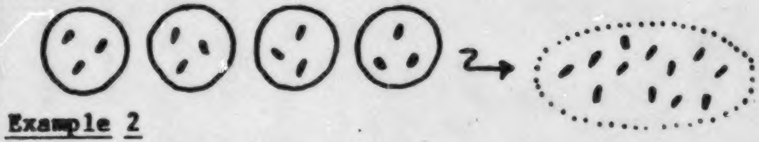
Not only do the principles underlying number and written number clash, but the principles underlying the arithmetic operations and written computations can clash as well. Multiplication provides us with an example of this clash (see Figure 8). In the first example of 3 x 4 (e.g., three pieces of candy on each of four plates), multiplication is the action of repeated addition, that is, adding groups of three discrete units four times. The notation uses a multiplier, 4, which represents the number of times that the action of addition (+3) has to be repeated. The result is twelve discrete units. The difference between addition (3 + 3 + 3 + 3) = 12 and multiplication (3 x 4) = 12 is that muliplication is an operation on an operation, or a second order operation: the new term, the multiplier, represents an operation (number of times) on an operation (addition of +3). The result (12 pieces of candy) is the same in both cases.

But Easley (1981) points out that multiplication is more wondrous than this analysis suggests, for the operation of multiplication seemingly enables one to create new entities, such as area or volume, out of other entities, such as line segments. Multiplication, in other words, can appear to be something more than repeated addition. In the second example (Figure 8), a horizontal line segment of 3 inches (a piece of chalk 3" long) is moved vertically down (a blackboard) a distance of 4 inches, thus creating a new entity which is no longer a line segment, but rather

Notation for the actions (operations) in both of the following examples:

$$3 \times 4 = 12 \quad \text{or} \quad \begin{array}{r} 3 \\ \underline{\times 4} \\ 12 \end{array}$$

Example 1


twelve candies in all

Example 2

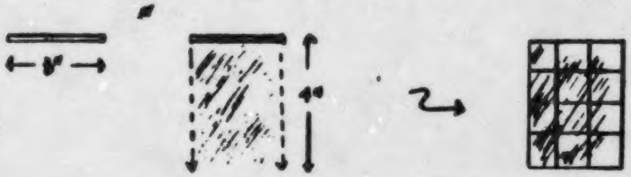

3"
4"
twelve square inches in all

Figure 8. Multiplication as simply repeated addition (Example 1), vs. multiplication as creating new entities (Example 2).

a "smear." Out of an operation on two line segments of 3 inches and 4 inches, a new entity is generated, and that is an area which is 12 square inches. The computational form and the written representation in this case are the same as in the example of the candy, nonetheless. A child who has been taught to think of multiplication as repeated operations of addition might be justly perplexed about how to conceive of the chalk smear as "repeated additions."

The foregoing analyses suggest that the construction of number concepts (necessary equality of two or more collections containing the same quantity of objects) and numerical part-whole relations (grouping of objects into sub-sets does not change the numerosity of the whole) rest in relationships that each person constructs. Numerical quantities can be represented symbolically (drawings or personally motivated means that emanate primarily from individual cognition), or they can be represented in conventional notation (digits and other notational devices that have a shared meaning). Conventional representational systems, acquired from the culture, enable us to record, store, and retrieve ideas in a common and accessible form. But the systems themselves have properties, and their objects (marks) are socially used to serve a wide range of functions. Notational systems thus allow for possibilities, and impose constraints, that are different from those encountered in symbolization.

The acquisition of the conventional system includes learning the digits (how to write them, and how they correspond to number-names). But more importantly it involves a reconstruction of the numerical and notational principles that organize the digits into our numeration system.

Place value is a most important property of the written system. As its name suggests, it is composed of two ideas. The first is the *notational* principle that written position designates numerical value. The second idea is the *numerical* principle that a power of ten is associated with each written position.

If a child's knowledge of number develops along with other aspects of his knowledge, so does his understanding of the objects for representing number. The general relationships that have been discussed thus far are diagrammed in Figure 9.

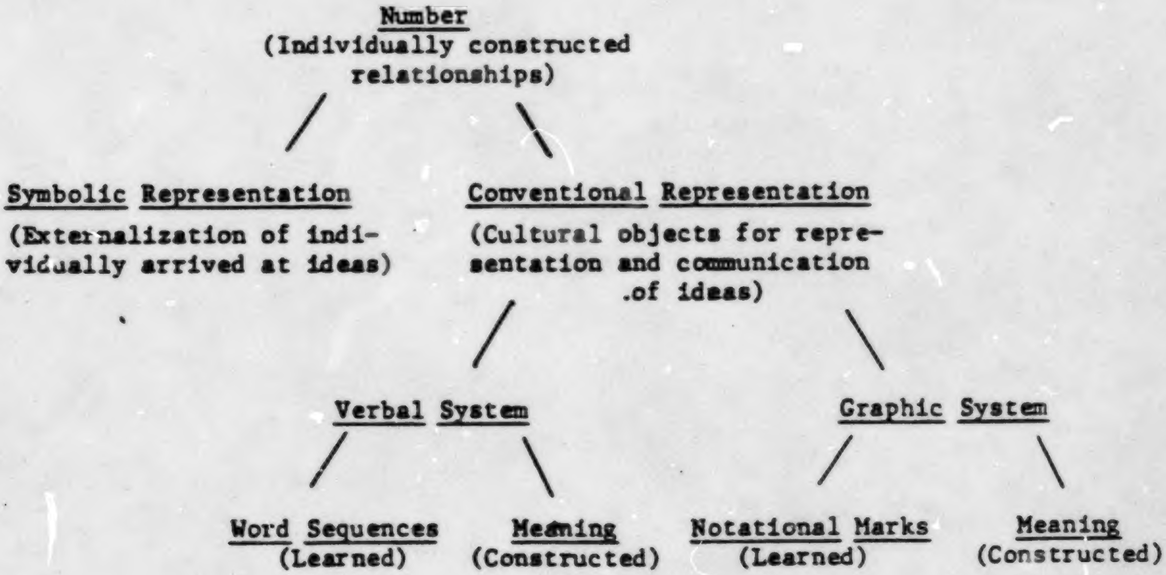

Number
(Individually constructed relationships)
Symbolic Representation
(Externalization of individually arrived at ideas)
Conventional Representation
(Cultural objects for representation and communication of ideas)
Verbal System
Graphic System
Word Sequences
(Learned)
Meaning
(Constructed)
Notational Marks
(Learned)
Meaning
(Constructed)

Figure 9. General relations between number and numerical representation

I suspected that the relations suggested in Figure 9 would be constructed by children over time, and that successive understandings in the process of construction would form developmental levels. In order to explore some of these relations, a set of hypotheses were devised that served as a frame of reference for the empirical work. In the following section the hypotheses are specified and explained.

Hypotheses

The hypotheses set forth below should be understood as heuristic premises that set the direction for the empirical work. They should be seen as giving direction to several heuristic lines of study, rather than as hypotheses that can be confirmed (the null hypothesis rejected). They have a methodological as well as substantive aspect, and they vary in the extent to which they reflect new, as opposed to previously investigated problem areas. The hypotheses are thus of different kinds.

The hypotheses are ordered in several interrelated ways: from premises that are more general to those dealing more specifically with the research problems studied here; from premises that are based on widely replicated findings to those that rest on relatively little empirical work; from premises that lend themselves primarily to qualitative results to those that do not; and from statements that are the least to the most susceptible of formal treatment. The first group of three hypotheses that posit development in three conceptually distinct domains have a different status than the group of ensuing three hypotheses (numbers 4, 5 and 6) that suggest more specific relationships between developmental lines among the domains, or the last hypothesis that deals specifically with the strength of a statistical relationship among two between-domain pairs. The reader will note that all of the hypotheses are subjected to the same kinds of formal analysis procedures in Chapter 6. The trade-off between methodological consistency and theoretical consistency, the latter of which would caution against uniform treatment, will become apparent. The results of the formal analysis, detailed in Chapter 6, bears most convincingly on the seventh hypothesis. Conversely, the results of the qualitative analysis, presented in Chapter 5, deals more directly with the first three hypotheses, less directly with the second group of three hypotheses, and not at all with the seventh. The results of the qualitative analysis that form the subject matter of Chapter V form the heart of the exploratory and descriptive results of this study, and not the quantitative results as a whole.

Some of the hypotheses, and the tasks associated with them (see Chapter IV), were taken directly from Piaget's well known and widely replicated work. The first hypothesis regarding conceptual development, and the conservation of elementary number that is associated with it, are the primary referents here (Piaget, 1941). Other hypotheses were derived from the work of Piaget's Genevan collaborators who extended various aspects of his many studies, or who opened new ground in genetic epistemological studies. Incorporated into these hypotheses are the results of studies bearing on development in the area of symbolizing numerical quantities (Sastre and Moreno, 1976), in the extension of number concepts (Greco and Morf, 1962), and in the meaning of conventional notational systems and graphic marks (Ferreiro and Teberovsky, 1979; Siegrist, A. Sinclair and H. Sinclair, research in progress). Still others were suggested by pilot work done by myself and carried out in the spirit of Genevan research. Yet others were suggested by sources outside the Genevan school.

The hypotheses are listed and explained below. To make them more amenable to formal quantitative analysis, they are reformulated as direct questions, with reference to specific tasks, in Chapter IV.

Hypothesis 1. There are levels of cognitive structuring of number and numerical part-whole relations.

Number is understood to be a general universal cognitive structure that is elaborated over time by means of coordinating ordering and inclusion relationships. While number in its full blown sense is atemporal and aspatial, its elaboration by individual children requires experience with, and thought about, objects and relationships in space and in time. The process of construction is gradual.

Young children must construct linkages between the many specific occasions on which they make correspondences, and impose order, on objects and events. Some of these linkages will result in number. Children as young as 18-24 months have been observed to engage in systematic behavior with objects which suggests one-to-one correspondence and serial ordering at the sensorimotor level (Moreno et al., 1976). This early behavior indicates that at the level of sensorimotor intelligence, children are engaging in actions that prefigure what will later be reconstructed at the level of thought. In the Piagetian view, one-to-one correspondence had to be synthesized with serial ordering, without regard to the qualitative properties of objects, in order for number in its real meaning to be understood.

Slightly older children have to construct the notion of "a unit" from many specific instances of "the unit" in order to understand the idea that forms the basis for enumeration (Steffe et al., 1981). Children spontaneously apply the counting word "two" to identical things (common objects that occur in pairs, such as shoes), to symmetrical body parts (e.g., feet or eyes), and to repeated actions (e.g., climbing stairs) before they apply these words to dissimilar objects and actions, or to similar events that are separated in time (e.g., two trips to the playground). Hence it is plausible that early on, children understand the counting words as lexical items that apply to similar objects or actions rather than as words that name quantities. From this point of view, a major conceptual hurdle is overcome when the child can differentiate between "one, two" as words which name similarity or identify, and as words that name amounts.

Studies of children's counting have shown that very young children can accurately count collections containing two or three items, and slightly older children can count sets made up of larger amounts (Gelman and Gallistel, 1978). But the child's precise and ordered application of a set of tags (words) may or may not imply that he has numerically quantified the collection. In addition to differentiating among many quantitative dimensions to abstract numerical quantity as referring to the cardinal value of a collection (e.g., length of a row of objects versus the number of elements it contains), the child has to understand the idea of a unit (Steffe _et al_., 1981). At what point in the sequence of counting or of structuring numerical quantities is the child able to "lift" the unit from its myriad particular instances and understand it as "a unit" (the basis for enumeration) rather than "the unit" (limited to specific collections of objects)?

It is likely that children grasp numerical quantities of two or three before understanding quantities of five or six, and that larger amounts (ten or twelve) take even longer to structure. Our numeration system uses the base ten, and from the perspective of the young child, this quantity is a huge amount. It is not reasonable to expect children to grasp the numerical principle of "one ten equals ten ones" (1 x 10 = 10 x 1) that is linked with place value understanding before they are able to grasp the idea of ten as a whole made up of ten units (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 10).

It is hypothesized that numerical part-whole relations are structured along with the understanding of whole amounts. Numerical bunching or chunking (the division of a whole into subgroups of differing amounts, such as grouping six objects into sub-collections of two and four elements each) is an instance of numerical part-whole relations (Payne and Rathmell, 1975). The quantity six, for example, can be broken down into two and four, and five and one, as well as three and three. The structuring of these parts into the numerical whole is hypothesized to take place along with the structuring of the cardinal number six. The combinations of numerical parts which together make up ten are many (one and nine, two and eight, etc., as well as two groups of five, five groups of two and so forth). Thus it should not be surprising that they take a longer time to learn.

Hypothesis 2. There are levels in children's ability to represent numerical quantities symbolically.

Children impose or work out ideas on objects before they can represent those ideas in a drawing. Therefore children's ability to structure objects into numerical groups at the level of action (grouping objects in reality) is expected to be in advance of their ability to represent those groupings in symbolic form (Sastre and Moreno, 1976; Allardice, 1977). It is hypothesized that there are discernible developmental levels in children's symbolic productions (Sastre and Moreno, 1976), and that these levels loosely parallel, and lag behind, levels of cognitive structuring. No expectation of identity between one child's drawing and another child's drawing of a given numerical quantity is implied here. In categorizing children's drawings of objects, both the relative difficulty of rendering various objects on paper, and the subject's experience with the drawing materials, has to be taken into account. It is nevertheless hoped that children's general level of symbolic performance can be assessed.

Hypothesis 3. There are levels in children's knowledge of the conventional notational system. Children's ability to write number-squiggles (digits) is distinct from their ideas regarding the meaning of those squiggles. Children's theories concerning meaning form a developmental sequence from varieties of non-quantitative to quantitative ideas.

Number-squiggles are objects of knowledge, and young children in our culture think about these squiggles from quite an early age (Ferreiro and Teberovsky, 1979). Very young children (from two-and-a-half years of age) are able to differentiate letter-squiggles from number-squiggles (H.Sinclair, 1980b). But the squiggles seem to have rather little meaning beyond being recognizable marks. By the time children have reached their fourth birthday, they have had numerous experiences with the squiggles and seem to have more systematic notions about them.

Numerals appear in the general environment on all sorts of objects: clothing; vehicles; houses; clocks; books; birthday cakes, etc. Not only are the host objects varied, but the functions which numerals serve are varied as well (prices, bus routes, sports scores, dates, to list but a few). It is presumed that children construct their own ideas about number-words and number-squiggles, as well as ideas concerning the linkages between the two (Siegrist _et_ _al_., in preparation). They build these notions as they hear, see and use them in conjunction with all sorts of activities. The differentiation of these linkages is crucial to children's understanding that number-squiggles and counting words can refer to the ideas of cardinal value rather than to specific objects, particular activities, or special events.

It is expected that digits representing small quantities will be given a numerical meaning prior to numerals representing larger amounts. Multi-digit numerals wherein numerical value is specified by the positional notation of digits (place value) should be a late construction. This acquisition should depend in part upon the capacity of the child to structure the relationships between the numerical parts and the numerical whole designated by the numeral (e.g., three and twenty and one hundred in the numeral, 123).

Hypothesis _4_. There is a positive relation between children's cognitive developmental level and level in symbolic representation.

It is hypothesized that there is a positive relation between level of cognitive structuring and level of personal graphic representation; children who perform at higher levels in tasks designed to tap the former will also be at higher levels in tasks devised to show the latter.

This hypothesis rests on the assumption that very general cognitive structures underlie and direct a child's behavior. Number is a general structure, and a child's symbolic representation is a manifestation of that structure. What a child draws is a reflection of what he or she thinks. The relationship between these domains ought to be closer than the relationship between cognition and conventional representation because the child can draw quantities in whatever way he or she wants. In addition the child is not constrained by having to know how to make particular kinds of shapes (graphic squiggles).

Hypothesis 5. There is no consistent relation between children's cognitive development and their knowledge of the notational system.

It is hypothesized that there is no consistent relation between children's cognitive level and their levels in conventional notation. On the one hand numerals are culturally given objects which must be learned. On the other hand the numeration system has specific properties that presumably must be reconstructed by individuals. A child who knows how to write squiggles does not necessarily understand the numerical significance of them. A child who does not know how to write squiggles does not necessarily lack knowledge of number (Piaget, 1941).

It is hypothesized that children who are further along in conceptual development will not necessarily be knowledgeable in conventional notation. Conversely, children who demonstrate proficiency in conventional notation will not necessarily have highly developed number concepts.

Hypothesis 6. There is no consistent relation between a child's ability to represent quantities symbolically and his/her ability to represent them in conventional notation.

If there is no consistent relation between cognitive structuring and conventional representation (Hypothesis 5), and if personal representation follows upon cognitive structuring (Hypothesis 4), then one would expect no consistent relation between levels of personal and conventional representation. If, on the other hand, some relation is found, then the results of hypotheses four and five above would have to be reexamined.

Hypothesis 7. The relation between conceptual development and personal representation will be stronger than the relation between personal and conventional representation.

This hypothesis concerns relations between relations, that is, the strength of the relation found in Hypothesis 4 against the strength of the relation found in Hypothesis 6. It is hypothesized that the relationship between cognitive level and symbolic (personal) representation will be closer than the relationship between symbolic and conventional representation. Children who have structured quantities as large as twenty-three, for example, may have difficulty interpreting the relation between the 2 in 23 and twenty objects. Similarly they may have difficulty in dealing with the meaning of zero.

These general hypotheses had to be translated into concrete situations (experimental procedures) in which children could work with materials, "play" with relationships, and talk about their ideas. These procedures or tasks are described in the next chapter which discusses the methods used in this exploratory and descriptive study. The reformulation of the hypotheses into simpler questions can also be found there.

CHAPTER IV

METHOD OF STUDY

In the psychogenetic method of study, theory-based hypotheses are explored in experimental situations devised to elicit children's ideas about the aspect of knowledge under investigation. Children are asked to use experimental materials to solve problems, draw inferences, make judgments, and explain their solutions or conclusions. This chapter opens with a description of the tasks used to explore the hypotheses described in Chapter III. The empirical work focused on children's reconstruction of the place value property of the notational system. To this end the tasks were designed to expose children's knowledge concerning place value, and the cognitive and symbolic representational abilities thought to be linked with place value understanding.

Next, the subjects to whom the tasks were given, and the situations in which they worked, are described. The design of the study was cross-sectional. Finally, the procedures used to analyze the data obtained from the interviews are discussed. These include both qualitative and quantitative procedures. The research was designed to uncover levels and test for relationships among levels in the domains of cognition, symbolic representation, and conventional representation. It was not designed to explain movement from one level to the next.

Task Descriptions

Three tasks were used to assess children's conceptual development in number, and these tasks are described first.

(1) Conservation of Elementary Number

Materials: A standard boxed collection of poker chips containing red, blue and white chips.

The Interviewer (I) asked the child (S) to choose which color chip he wanted to work with. I took chips of a different color and made a row of chips in front of S. (Younger children were asked to pretend that the chips were cookies.) I asked S to use his chips and "put out just as many, so that your row and my row will have just the same amount (to eat)." After S made his row, he was asked to watch because I was going to "do something to your row." I spread S's row into a spatially longer row and asked, "Does your row and my row still have the same amount (to eat)? How do you know that?" I then pushed S's row together into a heap and asked. "What about now? Do we (still) have the same amount (to eat)? Or does one of us have more? Why?"

In cases where S's responses were not clear, I made two additional transformations: split S's large heap into two smaller heaps; and stacked S's chips into a tower. In both cases, I asked, "And what about now?"

(2) Establishing Equality Between Unequal Collections

Materials: Two stuffed animals (a puppy and a lion cub that resembled a kitten) and poker chips of the same color that the child had chosen for the conservation task.

S was asked to choose one of the animals to work with, and I took the other. I placed six chips in front of S's animal and four chips in front of her animal. "Let's pretend that these are pieces of food. Do you think that it's fair for your (animal) to have that much to eat, and my (animal) to have this much to eat? Can you make it fair?" (For younger children, I elaborated this story by wailing, "oh, my kitty is so sad, he's crying 'cause he doesn't think it's fair ... he wants just as much for his dinner as the doggie got. Can you help him out?")

This task was used to observe how S went about establishing equality: by removing two chips from her animal's collection; by adding two chips from the box to the other animal's collection; by collecting all of the chips and redistributing them among the animals, one-by-one; or by simply removing one from her animal's collection and giving it to the other.

(3) Anticipating Equality between Unequal Collections without Counting

Materials: The two animals used above, and a large collection of square wooden blocks (1 1/4" x 1 1/4" x 1/4").

S's animal received a large quantity of blocks (n = about 25) and I's animal received a smaller amount (n = about 12). I asked S if one of the animals had more. Then S was asked to watch as I began removing blocks, one-by-one, from S's collection. After S's collection had dwindled into a clearly smaller heap than I's, S was asked, "Now who has more?" Then S was told that he was going to be asked a funny sort of question, so he had to listen carefully. "Do you think that there was one moment, just one time, when your animal had just the same amount as mine?"

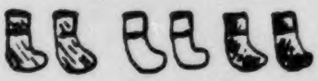
This task was designed to see whether S could infer the momentary equality between the collections, when he was prevented from counting the initial collections, and given no reason to count as the blocks were being removed from one of them.

Socks and Pairs (Task 4), Wheels and Cars (Task 6), and Packs of Gum (Task 7) were used to probe children's ability to deal with numerical quantities, both as ungrouped collections and as subsets of grouped objects.

(4) Socks and Pairs

Materials: Three pairs of toddlers' socks, one brown, one red, and one blue pair.

Six socks were laid out in front of S in a pattern suggestive of pairs:



"Look what I've got here, a whole bunch of socks. Can you figure out how many pairs of socks there are? Can you put them into pairs?" I then picked up each pair and folded the socks into a "ball" (in the fashion that many of us do before putting socks away in a drawer) so that one sock "disappeared" into the other. When all of the pairs were folded, S was asked, "How many socks do you think there are altogether? And how many pairs?"

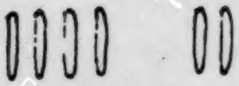
The interest of this task was to find out, in a familiar and concrete context, whether S was bothered by "six" becoming "three" and reverting back to "six."

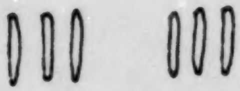
The next task required children to make three separate drawings of six objects, spatially arranged in three different part-whole relationships.

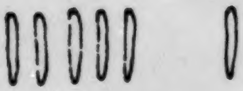
(5) Drawing Sticks

Materials: Six popsicle sticks, three sheets of paper (9" x 12") and a choice of drawing materials (crayons, craypas, and magic markers).

I used the six popsicle sticks to make the following arrangements in front of S:

(a)


(b)


(c)


After the first arrangement was made (four and two), S was given a sheet of paper and asked to make a drawing "to help you remember how the sticks were put out, because in a moment, I'm going to take that away and put the sticks out in another way." After S completed her drawing, I picked up the sticks and used them to make the second arrangement. S was given a second sheet of paper and asked to "make something to help you remember that." Then I picked up the sticks, made the third arrangement, and gave S a third sheet of paper. When S finished the drawing, I took away the sticks and lined up S's three drawings vertically. S was asked to look at her drawings and was asked, "Is there one drawing that has more than any other?"

This task was used to see how S would draw the subgroups, and whether S would compare the spatially separated parts, or the numerical wholes, as she evaluated each drawing of six sticks.

The next two tasks focused on the relation between children's grouping of objects, their symbolic representation of the action (drawings of units made into groups), their notation of quantities with numerals, and finally their ideas concerning the quantitative relation(s) between the numerals and objects rendered. Of greatest interest was what the child would make of the relation between individual digits and the quantities they represent (place value). The two tasks were parallel in structure but differed in the number of elements used, and in the set (group) sizes made with the elements. The objects (wheels and sticks of gum) were selected for a specific reason. They are familiar objects in the child's environment in both individual and grouped ways (a car needs four wheels, and packs of gum contain five sticks).

(6) Wheels and Cars

Materials: A car made out of Tinkertoys (wood and plastic modular pieces that can be put together to create different kinds of objects) with four removable wheels; a wooden cigar box containing twelve additional Tinkertoy wheels; paper (12" x 18") and drawing materials (the same assortment used for "Drawing Sticks" above).

I showed S a Tinkertoy car and asked, "How many wheels does a car need?" Then a box containing twelve more identical wheels was opened and the wheels dumped out in front of him. The wheels from the car were also removed, thus making sixteen wheels in all. "Can you figure out a way to tell how many cars we could outfit with all those wheels?" Note was taken of what S did with the wheels (e.g., line them up, or make particular kinds of patterns as he composed the sets of four, etc.). Then S was given a large sheet of paper and the drawing materials, and was asked to make a drawing of the wheels "so that if someone else came along and looked at your drawing, they could tell that we had all those wheels, and we could make (four) cars with them." After S drew the wheels, he was asked to "write the number for how many wheels you drew in all" and then "for how many cars we could make."

Using a different color marker for each digit, I drew circles, first around the 6 and then around the 1 in 16, and asked after each circle, "Do you think this part of your sixteen has anything to do with the amount of wheels that you've drawn here? Can you take this marker and show me in your drawing?" Then I drew circles around the whole 16 and finally the 4, asking each time, "Do you think that has anything to do with what you've drawn?" Care was taken not to call the 1 in 16 "one" or the 6 in 16 "six." The wording, "that part of your sixteen" was strictly observed.

(7) Packs of Gum

Materials: One pack of gum with the top torn off (so that the five sticks were visible); twenty-three loose sticks of gum; paper and drawing materials (as used for "Wheels" above).

S was asked to verify that a full pack of gum contained five sticks. Then S was given the twenty-three loose sticks and asked if she could figure out "how many packs of gum I had to open up to get all of that gum there." Note was taken of S's grouping action, counting strategy, and what she did with the "remainder of three." S was again given paper and drawing materials and asked to make a drawing of what she had done. The rest of the procedure was identical to that used in "Wheels" above. After S had completed the drawing and had written the numerals, she was asked to interpret the meaning of the digits and numerals with respect to the symbolized quantities. Again, the wording "that part of your twenty-three" rather than "that two" or "that three" was strictly observed.

The next task was used as a check on the children's notions about the meaning of the digits as they had shown them on the "Wheels" and "Gum" tasks above. For this task, I took over the job of doing the drawing and writing.

(8) Other Digits and Numerals

Materials: Paper and drawing materials (as used in "Wheels" and "Gum" above).

I told S, "You've done all of the work so far, and I'm sure you're getting tired. This time I'll make the drawings." Where appropriate, I added, "They won't be nearly as complicated as yours, just really simple ones. I'm not much of an artist." Then I drew the following quantities of marks (beginning with n = 6 for the younger children and n = 14 for the older children), followed by the respective numerals. S was asked each time to indicate what the individual digits ("that part of the fourteen," "that part of the twenty," etc.) and the whole numeral had to do with the quantity of marks made.

(a) six lines, and the numeral 6;

(b) fourteen X's or lines, and the numeral 14;

(c) twenty circles, and the numeral 20;

(d) five more circles, and the numeral 25;

(e) one hundred and five lines, and the numeral 105;

(f) five of the above lines were covered up, and the numeral 100.

The above quantities were chosen with specific ideas in mind: (a) six is a single-digit number; (b) fourteen is a two-digit number involving a "teen" which is conducive to "reversal"; (c) twenty is a two-digit number necessitating the end-point use of zero; (d) twenty-five is a two-digit number, not involving a "teen," and not involving zero; (e) one hundred and five is a three-digit number using zero in a medial position; and (f) one hundred is a three-digit number using zero in both medial and end positions.

The last task that some of the children did was a marble game. For reasons of time as well as safety, none of the four- and only some of the five-year olds played the game. Among the other age groups, most of the children played and enjoyed the game. Originally the game was designed to elicit children's symbolic representation of quantities (keeping score without using numbers), with a special eye towards how they symbolized zero.

(9) Marbles

Materials: An 18" square marble board, covered with green felt, on which a circle made of yellow and white rick-rack was glued; a large quantity of marbles, including one large "shooter marble"; paper and pencil.

To begin the game, eighteen marbles were placed in the middle of the circle. The child was told that the object of the game would be to knock out as many marbles as possible, using the large chooter marble. S and I took turns, and S was asked to keep score: "Could you think of a way of keeping score without using numbers? Any way at all, only not using numbers."

Subjects

Eighty children between the ages of 4:2 and 9:9 from urban and suburban middle class backgrounds were interviewed individually, either in their homes or at their schools, by a single Interviewer. All of the children were "normal," cooperative, delightful individuals. None of them had any physical or mental handicaps, and none of them faced any unusual environmental deficits either at home or at school. Most of the children liked arithmetic; only a few remarked that "they weren't too good with numbers" or "didn't do very well in math" in school.

Most of the youngest subjects (four- to five-and-a-half-year-olds) attended a university day care center in Cambridge (Radcliffe Child Care Center). The vast majority of children above the age of five-and-a-half were recruited from a single neighborhood elementary school in Belmont (Winn Brook School). The remainder of the children were participants in a summer day camp program at a private school in Cambridge (Buckingham, Browne and Nichols School), or were friends and acquaintances of the Interviewer.

The only criteria that were exercised in the selection of subjects, beyond their being normal, middle class children, were (1) the child's willingness to be interviewed, and (2) parental permission to interview the child. Admittedly there is a kind of self-selection among the pool of subjects who participated in this study. The numbers of children in each age group are given in Table 1. The first column gives the number of four-year-olds, five-year-olds, etc., and the second column reflects the number of children when two age groups are collapsed.

Age	n	n
4:2 - 4:11	12	27
5:0 - 5:11	15	
6:0 - 6:11	12	29
7:0 - 7:11	17	
8:0 - 8:11	12	24
9:0 - 9:9	12	
TOTAL	80	80

Table 1. Number of subjects in each age group

The distribution of subjects by age and sex is given in Table 2. The number of males and females are once again given by separate and collapsed age groups.

Age	Males		Females	
4:2 - 4:11	8	16	4	11
5:0 - 5:11	8		7	
6:0 - 6:11	8	13	4	16
7:0 - 7:11	5		12	
8:0 - 8:11	3	9	9	15
9:0 - 9:9	6		6	
TOTAL	38	38	42	42

Table 2. Number of males and females in each age group

The distribution of subjects by grade level is given in Table 3. Again the number of children in each grade level, as well as the number of children in collapsed grade levels, is given.

Age	Preschool, Kindergarten	First Grade	Second Grade	Third Grade	Fourth Grade
4:2 - 4:11	12				
5:0 - 5:11	15				
6:0 - 6:11	5	7			
7:0 - 7:11		9	8		
8:0 - 8:11				12	
9:0 - 9:9				5	7
TOTAL	32	16	8	17	7
TOTAL	32	24		24	

Table 3. Distribution of subjects by grade level

Of the 80 children who were interviewed, 27 were four or five years old, 29 were six or seven years old, and 24 were eight or nine years old. Of the 80 children, 32 were in preschool or kindergarten, 24 were in first or second grade, and 24 were in third or fourth grade. Overall 38 males and 42 females took part in the study. However the balance between males and females among age/grade groups shifted: among the 39 four- to six-year-olds, 24 were boys while 15 were girls; among the 41 seven- to nine-year-olds, 27 were girls while 14 were boys. Gender was not thought to be an important variable in this study; therefore no effort was made to obtain a more balanced distribution of boys and girls.

Design of the Study

This descriptive study, while exploratory in content, has features in common with more standard experimental work. From a purely experimental point of view, this study is closer to the pilot testing phase of research than to experimentation. From a Piagetian perspective, however, such exploration is central to discovering if there are levels in children's natural construction of knowledge in a particular domain, that is, to determining whether the sequence of construction is developmental, and, to describing the levels that are found.

In experimental studies, hypotheses are articulated in terms of a finite number of discrete variables that are thought to cause or contribute to a particular outcome, singularly or in interaction. Because this study calls for the discovery and description of developmental levels in two relatively uncharted areas (symbolic representation of quantities, and knowledge of the notational system), the hypotheses are very general and were given in propositional form. Propositions lend themselves to categorical evaluation: either there are, or there are not, levels of development in the areas being studied.

In designing an experiment to test for the effect of isolated variables on an outcome, as many variables as possible are controlled in order to isolate the effect of the hypothesized variables (or in an experimental intervention, the effect of the specific treatment) on the observed outcome. Adherence to a strict set of experimental procedures is important from the standpoint of being able to replicate the experiment. An exploratory study such as the one undertaken here does not ignore the need to control variables; nor does it disregard the need for specifying and carrying through with a set of procedures. But neither are regulated to the degree that they are in a strictly experimental design.

First, the clinical interview technique of free conversation with children was employed. Where unexpected ideas arose from children, the Interviewer followed them through with probes (e.g., "Hmmm, let me see if I understood that — do you mean...?" "Oh, that's an interesting idea, can you tell me a bit more about that? Did you learn that from somewhere? Did someone tell you about that"). If a child gave conflicting accounts in two tasks (e.g., different interpretations of the quantitative meaning of each digit in a two-digit numeral), the Interviewer went back over the tasks with him to see whether the apparent difference mattered enough to motivate him to change his mind. In other words, when the child was made aware of the difference between his two accounts, did that awareness lead to his repudiation of one of the ideas, or was his level of understanding such that the inconsistency made no difference to him? In addition, when the Interviewer sensed a resistance on the part of the child to finishing a task, she abandoned it and went on to the next task so as to prevent the "loss" of the child to frustration, boredom or fatigue.

Second, the order in which the tasks were given was not counterbalanced; it was the same for all subjects. The cognitive tasks (Tasks

1 through 4) were followed with the tasks that called for symbolic and conventional representation (Tasks 5 through 7). Task 8 where the Interviewer made the representations, and where no objects were involved, was done next, and the marble game (Task 9) concluded the session.

Third, as the study was not an experimental intervention, no pretest or post-test measures of any kind were administered, and no control group was established. Care was taken to have a sufficient number of children within each two-year age grouping (four- and five-year-olds, six- and seven-year-olds, and eight- and nine-year-olds) to ensure a representative sampling of middle class children's ideas in this age span, and to permit some statistical analyses of the data.

In experimental studies, formal descriptions of phenomena are generally stated in probabilistic terms, for this allows one to make predictions based on the regularity that was observed in the experimental situation. In this study, such formal descriptions are few.

Finally, the situations in which the children were interviewed were not the same for all subjects. Among the younger subjects, most were interviewed in a school setting, while the rest were visited in their homes. Among the older children, the situation was reversed: most were interviewed in their homes, while only a few were interviewed at school. The only determinants of home versus school setting were time of year (some subjects were seen during vacation periods) and parents' judgments regarding where their child would be most comfortable being interviewed. It should be added that in each of the school settings, the teachers and administrators were very supportive of this research and gave the Interviewer a separate, quiet room in which to conduct the sessions.

Each child was asked to do many tasks, and thus the interview was long. The younger subjects were seen twice for sessions that lasted from fifteen to twenty minutes each. The older subjects completed the tasks in a single session that took anywhere from forty to ninety minutes.

All of the interviews were tape recorded and transcribed by a single individual, the Interviewer. Most of the original drawings are in the possession of the Interviewer. The children who wanted to keep their drawings were asked to lend them overnight so that they could be duplicated. In all cases the drawings of these children were returned to them within a day or two. A sampling of the children's drawings can be found in Appendix A.

Data Analysis Procedures

The hypotheses which we want the analysis to address are given below, reformulated as direct questions. The hypotheses/questions and the tasks (described above) were paired as shown in Figure 10. Hypotheses 1 through 3 concern levels, and hypotheses 4 through 7 concern relations among levels.

Hypothesis/Questions	Tasks
1. Are there levels in conceptual development? in the ability to structure a collection of objects into subsets?	Number concept: Tasks 1, 2 and 3; Grouping objects into subsets: Task 4 and actions in Tasks 6 and 7.
2. Are there levels in representing numerical quantities symbolically?	Tasks 5 and 9, and drawings in Tasks 6 and 7.
3. Are there levels in reconstructing the notational system? in grasping place value?	Writing and interpreting number-squiggles in Tasks 6, 7 and 8.
4. What is the relationship between levels in cognitive development and levels in symbolic representation?	Cognitive levels (results from tasks associated with Hypothesis 1) against symbolic representation levels (results from tasks linked with Hypothesis 2).
5. What is the relationship between levels in cognitive development and levels in conventional representation?	Cognitive levels (results from Hypothesis 1) against notation levels (results from Hypothesis 3).
6. What is the relationship between levels in symbolic representation and levels in conventional representation?	Symbolic representation levels (results from Hypothesis 2) against notation levels (results from Hypothesis 3).
7. Is there a stronger relationship between levels in cognitive development and symbolic representation than between levels in symbolic representation and conventional representation?	Strength of Hypothesis 4 relationship against strength of Hypothesis 6 relationship:

Figure 10. Relations between hypotheses/questions and tasks

Two forms of data analysis are required to address these seven hypotheses/questions: qualitative, and formal or quantitative methods. Qualitative analyses are used to establish levels of knowledge or behavior within each of the tasks. To discern within-task developmental levels, the actions, explanations, and graphic productions of each child on each task are initially noted. Depending upon the focus of the task, the relevant data consist of actions while manipulating objects, explanations of judgments rendered, procedures for representing ideas in drawings, and modes of writing and assigning meaning to numerals. The children's responses are then compared, both within and across age groupings, for patterns which suggest developmental differences.

Ideally, younger children will act in terms of one constellation of ideas, while slightly older children will use another, and still older children yet another. Each of these constellations should reflect progressively more adequate conceptualizations of the task. To the extent that such constellations emerge in an orderly (roughly age-related) way, they can be treated as within task developmental levels. Once levels have been established, contingency tables are constructed for formal analysis. A level x age table is made up for each task, and children's names are placed in their appropriate cells. The Chi square statistic is used to test for independence, and Cramer's statistic is used as a measure of the association found. The results of these procedures are reported in Chapter V.

In the next phase of the analysis, the questions raised as Hypotheses 1, 2, and 3 are formally addressed: are there developmental levels in each of the three separate domains? The answers to these questions are derived from making a series of comparisons among children's performance on the tasks that were assigned to tap their abilities in the respective domains: the cognitive tasks linked with Hypothesis 1; the symbolic representational tasks associated with Hypothesis 2; and the conventional representational tasks specified with Hypothesis 3 (see Figure 10). If similarities in performance among the tasks in each of the domains is found, then those results can be interpreted as reflecting within-domain developmental levels. They can be treated as having tapped similar, or highly related, developmental phenomena.

A series of contingency tables are constructed in which performance levels among all combinations of tasks within each of the domains can be compared. For example, the data for cognitive development include children's performance on number concept tasks (Tasks 1, 2 and 3) and on grouping tasks (Task 4 and actions in Tasks 6 and 7). Thus the tables are composed of levels in Task 1 x levels in Task 2, levels in Task 1 x levels in Task 3, levels in Task 3 x levels in Task 3, and so forth. Chi square and Cramer's statistics are performed on each of these tables.

Once the within-task and within-domain results have been established, one is positioned to address the questions listed as Hypotheses 4, 5, and 6: are there relationships among developmental levels in the cognitive, symbolic representational, and conventional notational domains? The series of comparisons that these hypotheses necessitate are between-domain levels of performance: number concept tasks x symbolic representational tasks; conceptual tasks x conventional representational tasks, and symbolic x conventional representational tasks. Chi square and Cramer's statistics are applied to each of the tables composing this set of data as well.

Finally, the relative strength of the relation between cognitive and symbolic representational abilities, and between symbolic and conventional representational abilities, can be deduced (Hypothesis 7). The results of the statistical analyses bearing on the seven hypotheses can be found in Chapter VI.

CHAPTER V

RESULTS: DEVELOPMENTAL LEVELS

The goals of this study were (1) to identify developmental levels for each of the tasks in the theoretically distinct domains of knowledge, and (2) to see whether a set of hypothesized relationships could be found among these levels, as reflected in individual children's performance on the tasks. The data analysis procedures used to address these goals were different. For the sake of clarity, the results are presented separately.

The first category of results concern within-task developmental levels that were found by means of qualitative analyses of the data. A summary of the levels can be found in Appendix B. The results are formally described in a series of level x age contingency tables that constitute Appendix C.

The second category consists of the results bearing upon each of the hypothesized relationships outlined in Chapter III. Hypotheses 1, 2 and 3 focused on within-domain developmental levels; hypotheses 4, 5 and 6 concerned statistical relationships between domains; and hypothesis 7 suggested that the statistical relationship between cognitive development and symbolic representation would be stronger than the relationship found between symbolic and conventional representation. These results are described in the tables that form Appendix D and E.

The results of the procedures used to discern developmental levels are discussed in this chapter. The discussion of the statistical results bearing on each of the seven hypotheses is reserved for Chapter VI.

A few words regarding the process by which within-task developmental levels were discerned will help to guide the reader in the following discussion. Structural analysis presupposes that generalized actions undergird and direct specific actions; conversely, specific actions are manifestations of broader developmental processes. Thus qualitative analyses that aim to describe levels, that is, to show general sequences of development, entail a selective focus upon certain actions from among the stream of individuals' actions, namely, a focus upon ideas and behavior that differentiate "qualities of thinking and knowing" of one group of children from those that characterize developmentally different (usually younger or older) groups of children. It is in this sense that the selected actions share features or properties in common that serve to define "stages" of development. Different kinds of stages have been shown to mark the course of construction in other areas of knowledge building.

Such qualities of thinking are necessarily general. In focusing upon them, ideas and behavior that oftentimes occur in unique combinations in individual children are relegated to a secondary place. Put differently, the analyst treats general qualities of thinking as "theme," and individual differences as "variation," and consigns the latter to a position of lesser importance. From the standpoint of describing the process by which individual children develop their understanding of number and numerical representation, this can be problematic.

For example, many children between the ages of five and eight interviewed in this study used two or more qualitatively different ideas in a juxtaposed or side-by-side fashion, especially in response to questions concerning the meaning of digits and numerals. The combinations of ideas that were used (notably in Tasks 6d, 7d and 8) made it difficult to categorize these children into levels that approached a useful degree of specificity.

The levels described in this study, and summarized in Appendix B, represent something of a compromise between the two descriptive goals of compiling a record of children's specific ideas and behavior on the one hand, and describing the developmental course of their knowledge building of number and numerical representation on the other. The levels are given in general terms: they signify the order in which new abilities emerge in children; and they form a logical sequence. The specific notions that children used are rendered as "types" within levels and reflect the verbal responses that children gave. Even at that, some children used ideas from two or three levels as they worked through Tasks 6d, 7d and 8 in particular.

Cognitive Development: Number and Grouping

Task 1. Conservation of Elementary Number (Piaget, 1941).

Level 1. The child does not establish equivalence between two rows of objects.

Level 2. The child establishes equivalence between two rows, either by one-to-one correspondence or by counting. However the child does not conserve.

Level 3. The child establishes equivalence between the two rows but vacillates between conserving and not conserving number. The child is inconsistent.

Level 4. The child conserves number unequivocally.

The levels for this task are well established and require no further elaboration. Fifty-eight percent of the four-year-olds were at Level 1 and 60% of the five-year-olds achieved Level 2. When these two age groups and levels are combined, they account for 74% of the responses, that is, 74% of the four- and five-year-olds could not conserve number. In contrast, all but a single subject among the six- through nine-year-olds (98%) conserved number unequivocally. Among the subjects interviewed for this study, therefore, conservation was achieved by the age of six.

Task 2. Establishing Equality among Unequal Collections.

Level 1. The child uses some non-numerical idea as the basis for making unequal collections "fair."

Example: the child pushes the two collections together and says, "They (the animals) have to share."

Level 2. The child has an intuition that the collections should be the same. But the child's notion is global; he has not differentiated between same numbers (numerical equality), same appearance (spatial arrangement), etc.

Example: the child says the collections have to be the same, but he does not make them equal in number; or he makes the collections equal but cannot explain why the result is "fair."

Level 3. The child thinks that equality can be established in one or two ways: by adding two elements to the smaller collection B (thus A = 6 and B = 6), and/or by removing two elements from the larger collection A (A = 4 and B = 4).

Level 4. The child establishes equality between the collections by moving one element from Collection A to Collection B, thus making two collections of five elements each. The child may do this directly or he may have first made collections of four and/or six objects each.

The levels in this task reflect the following sequence of ideas. At Level 1, a non-numerical or qualitative notion (e.g., sharing) forms the basis of judgment. At Level 2 some intuition of numerical equality is present, but it is not differentiated from other, non-numerical considerations (e.g., spatial arrangement of the object). At Level 3 the child believes that equality can be achieved by adding two objects to the smaller collection and/or by removing two objects from the larger collection. At the highest level, the child understands that equality can be made in one of several ways, at least one of which involves moving one element from the larger collection to the smaller one.

This task was easier for the five-year-olds than the first task had been. Eighty-seven percent of the five-year-olds were categorized as Level 3 or 4, in contrast to conservation where only 40% attained Level 3 or 4. The largest contributing factor to the relative ease with which they handled this task seemed to be the development of counting as a useful tool for quantifying small collections. It will be recalled that the conservation task uses a larger number of elements (a minimum of eight objects in each row). As was true for conservation, all but one of the six- through nine-year-olds achieved the highest level.

Task 3. Anticipating Equality between Unequal Collections without Counting.

A third task had been planned (and was in fact administered) to tap conceptual development in number. However because of some problems in the administration of that task, the results were not included in this study. As originally conceived, the task had been designed to see whether children could infer a necessary equality between two collections which, at the outset were unequal in one way, at the end were unequal in the opposite direction, but which were, of necessity, equal at one moment in time.

A large collection of blocks (n = approximately 25) had been placed in front of one animal, and a smaller collection (n = approximately 12) had been placed in front of a second animal. The latter collection remained untouched during the task. After the child had agreed that one collection was larger than the other, the Interviewer began removing blocks from the larger collection, one-by-one, until the collection which had initially been the larger, was clearly the smaller of the two. The child was then asked whether there had been "just one moment, just one time" when the altered collection had had "just as many" as the untouched collection. The thrust of the task was to see whether the child could infer that one could not go from "more" to "less" without passing through "the same" or numerical equality.

Two problems arose in the administration of this task. The first had to do with the physical set-up of the task. The Interviewer placed the removed blocks within easy reach of the child and thus enabled him to act on his temptation to count the blocks. Once the child had begun to count, it was difficult to keep him from continuing with it. The second problem involved a more serious shortcoming. For children who seemed curious but uncertain, the Interviewer did not have a second, related task prepared which (a) might have enabled the child to explore the problem in another way, and (b) might have provided the Interviewer with a better means of probing for nuances in children's thinking. When the child seemed perplexed, there was little way for him or her to go beyond, "I'm not sure...: or "I think so, but..." By the time these difficulties became apparent, it was too late to change the procedure and justify the inclusion of the results in this study.

The next time this task is given, the following experiment is suggested as a possible accompaniment (taken from Morf, in Greco and Morf, 1962). A collection of objects, either odd or even in number, is placed on a straight-edge on a table. The objects are then dropped from the straight-edge to the floor, one-by-one, until the collection on the floor is clearly larger than the collection remaining on the table. The child is then asked if there had been one moment in time when the collections had had exactly the same amounts. This task would allow the child to think through the numerical relations in a slightly different way.

In the meantime, the implications of the loss of the data from Task 3 are serious, for the portion of this study which deals with conceptual development remains incomplete without them. There are no data which specifically address children's conceptual development beyond the ages of six or seven, and we therefore have few clues about the relation between conceptual development and differences found in older children's responses regarding the meaning of numerals (understanding the notational system).

Task 4. Socks and Pairs

Level 1. The child counts the whole collection of individual socks, "one, two, three, four, five, six," or "one, two; one, two; one, two." But questions concerning "how many socks vs. how many pairs" are answered with a blank look, or talk about some other topic.

Level 2. The child treats the term "socks" and "pairs" as if they were synonyms. She uses the same number name to identify each.

Example: the child says there are "two, two, two"; "three pairs and three socks," or "six pairs and six socks."

Level 3. The child counts the socks (six), counts the pairs (three), and maintains the idea that there are six socks at the same time as there are three pairs.

Socks and Pairs was the simplest of three tasks in which children were asked to treat individual objects as units and as members of subsets that made up the same numerical whole (Grouping Wheels and Grouping Gum are the other two tasks). A collection of six socks can be considered as an instance of the quantity six; when the socks are paired (subsets of two) the collection can be thought of as three intermediary units. The levels in this task reflect (1) knowledge that a "how many" question can be answered by counting; (2) certainty that the questions, "how many socks" and "how many pairs" refer to the same thing; and (3) conviction that "three pairs" and "six socks" refer to the same numerical whole.

Among the four-year-olds, 27% gave Level 1 responses and 64% were categorized at Level 2. Among the five-year-olds, the percentage of responses shifted to 8% and 77%, respectively. In contrast to these younger children, all but a single child in the other age groups (six through nine) gave Level 3 answers. By the age of six, therefore, the children interviewed for this study had no problem grouping six objects into subsets of two, and used two different quantifiers to name the same collection.

Task 6a. Grouping Wheels.

Level 1. The child counts the wheels on the toy car, "one, two, three, four." The collection of twelve wheels, however, remain as separate objects and are not grouped at all.

Example: the child "counts" the total number of wheels (i.e., counts them imprecisely), and initiates "a new game" with the wheels. He makes no effort to group the wheels into sets.

Level 2. The child groups the wheels for one set (car) out of the ungrouped wheels, but leaves the remaining objects ungrouped.

Level 3. The child groups all of the elements into sets, but the set size is wrong.

Note: these children made sets of two, rather than four, wheels. The source of this error seemed to be the child's image of a car from its "side

> view" where only two wheels are visible. Curiously enough, these children did not give up their idea of "twos" despite the fact that a toy car with four wheels had been used just a few moments before.

Level 4. The child groups all of the objects into numerically correct sets (four wheels per car).

This portion of the Wheels and Cars task involved children's ability to structure a total of sixteen wheels into sets of four. It will be recalled that the children were first shown a completed Tinkertoy car. Its four wheels were subsequently removed and added to the collection of twelve additional wheels. Level 1 included those subjects who counted the four wheels on the ready-made car, but who did nothing with the additional twelve wheels. Level 2 children grouped four of the twelve wheels (i.e., "replicated" the model) but stopped after they had made a single set. Level 3 children grouped all of the wheels but insisted on making collections of two rather than four wheels, even though they knew that a car needed four wheels. Level 4 subjects grouped all of the objects into sets of four.

Children's performance in this grouping task produced some surprising results. While half of the four-year-olds were at Level 1, one-third of the group attained Level 4! Among the five-year-olds, the Level 4 responses doubled, relative to the youngest children, to two-thirds of the total responses. Ninety percent of the six- and seven-year-olds reached Level 4, and as would be expected, 100% of the eight- and nine-year-olds were at Level 4.

The relative ease with which the younger children grouped as many as sixteen objects into sets of four was unexpected. Even the younger children could, at the level of action, distribute objects "by fours."

But lest we impute too much significance to their ability to deal with n = 16 or assume that they can "divide by 4," let us go on to consider the results of the next grouping task, for the results of that task serve as a corrective to any such notions.

Task 7a. Grouping Gum.

Level 1. The child counts the gum in the opened pack, "one, two, three, four, five." The collection of thirteen sticks of gum, however, remain as separate objects and are not grouped at all.

Level 2. The child groups the gum for one set (pack) out of the thirteen sticks, but leaves the remaining sticks ungrouped.

Level 3. The child groups the gum for more than one, but not all, of the packs that can be made with the twenty-three sticks of gum.

Level 4. The child groups the sticks of gum into four packs and indicates that there are "not enough" for a fifth pack. Some of these children call the remainders "half a pack."

The gum task was similar to the wheels task in that the children were asked to make units (sticks of gum) into packs (groups of five). But the tasks were different in several ways: (1) the number of sticks were deliberately chosen to yield "left-overs" or "remainders" (n = 13 for the four-year-olds and some of the children aged five, and n = 23 for the rest of the subjects); (2) the groups were made up of five rather than four objects; (3) the physical, empirical meaning of four wheels was undoubtedly easier to grasp; and (4) the "grouping" question, for the older children, was phrased to suggest an action completed in the past rather than one that was henceforth to be done. In the wheels task, the child was asked, "How many cars can we make with all of those wheels?" In the gum task, she was asked, "How many packs did I have to open to get all of that gum?" It should be noted that for the four- and five-year olds, the question was in fact changed to "Can you figure out how many packs we could make with that much gum?"

These differences are highlighted because among the four- and five-year-olds, the results of the gum task were quite different from those in the wheels task. First, the number of four- and five-year-olds who even attempted the gum task declined by 33% and 27%, respectively; 8 rather than 12 four-year-olds, and 11 rather than 15 five-year-olds, tried the gum task. Of the four-year-olds who did the task, none reached Levels 3 or 4 (even with the present-tense form of the grouping question). Among the five-year-olds, 18% were scored at Level 3 and 27% achieved Level 4.

These results present a sharp contrast with the abilities demonstrated in Task 6a where 33% of the children aged four, and 67% of the children aged five, reached the highest level. The four factors outlined above (dealing with remainders; group sizes of five rahter than four objects; greater accessibility of the notion of "wheels for cars" versus sticks and packs; and questions about unseen actions), either singly or in combination, most likely accounts for the difference in group performance. But from the data available, there is no way to tell which one.

Developmental Levels in Symbolic Representation

Task 5. Drawing Sticks.

Level 1. The child makes some kind of a drawing.

Type A. The child draws something irrelevant to the task, such as a drawing of a house, a person, or an animal.

Type B. The child makes a single drawing of a collection of six sticks. This drawing may or may not show the spatial separation between the sub-collections of sticks.

Level 2. The child makes three separate drawings for the three arrangements of sticks.

Type A. The child shifts the mode of representation within or among the drawings, thus creating a mixture of symbols and signs. For example, collection A is represented with a drawing of four sticks and two sticks with a space separateing the sub-collections; collection B is represented with a single numeral, 6; and collection C is represented with two numerals, 5 and 1.

Type B. The child produces drawings in which the sub-collections are ambiguous, and thus it is difficult for him to use the drawings to accurately reproduce the sub-collections.

Note: the child uses one or some combination of the following means to show sub-collections -- spatial separation, change of color, and making boundary figures different in size from the rest.

Level 3. The child draws the correct number of sticks (wholes), and the sub-collections (parts) are clearly indicated. However the child compares the sub-collections (parts) within and among the drawings, rather than the wholes. Thus he says that five sticks and four sticks are more than one, two, or three sticks.

Level 4. The child draws each of the collections (wholes) and sub-collections (parts) accurately. He compares the drawings and says that none shows more than any other. "They're all the same." "They all have six."

Asking children to draw six sticks arranged in different ways (four and two, three and three, and five and one) was originally conceived as a means of getting them to (1) symbolically represent the spatially separated parts in the respective arrays, and (2) compare the quantities they had drawn. This was the first of three tasks in which children were asked to make drawings, and some unanticipated concerns emerged.

For some of the children, the drawings themselves became the focus of attention; making "good" replicas of the sticks, and later "good" drawings of wheels, cars, sticks and packs of gum, became as important as thinking through the quantitative questions. This was manifested in the younger children (four- and five-year-olds) by an insistence on tracing around the popsicle sticks, and other objects, rather than drawing them free-hand. Tracing is one technique for reproducing figures precisely, and adults, too, rely on it for tasks that are more demanding than what the hand and eye alone can achieve (e.g., making a map of the United States). In retrospect the choice of a circular object rather than popsicle sticks would have been better from the standpoint of ease of drawing for this younger age group.

Among the older children (eight- and nine-year-olds) these concerns became apparent in the sheer complexity of the drawings they made, and in the care with which they attended to details (e.g., colored lights atop a police car in Task 6b, or the lettering on the gum wrapper in Task 7b). In the six sticks task, a surprising number of older children made the task much more complex than had been intended. They pondered long and hard over "what can I make that as four and two," and made elaborate drawings, as can be seen in Appendix A.

The levels in Task 5 reflect both the symbolic representation and the comparison components of the task. Level 1 children made some kind of a drawing but did not represent the first arrangement, and did not go on to make a drawing of the next arrangement of sticks. Therefore the comparison question ("Does one of the drawings have more than any of the others?") could not be asked. Level 2 children did make three separate drawings but used a variety of means to represent the sub-collections, including a mixture of symbols and signs, and a distinguishing feature (e.g., length of a line) to make a boundary figure different from the rest. In contrast to Level 1 children, it was possible to ask these children the comparison question. Level 3 children made accurate replicas of the sub-collections but compared separate sub-collections rather than wholes; and Level 4 subjects compared the respective wholes. A representative sample of the children's drawings will be found in Appendix A.

Fifty-eight percent of the four-year-olds drew in a Level 1 fashion, and 25% of them drew in a Level 2 way. Among the five-year-olds, the distribution was 31% in each of the first two levels, but another 31% were scored at the highest level. The six-year-olds were distributed across Levels 2, 3 and 4, with 22% at Level 2, 33% at Level 3, and 44% at Level 4. Of the 25 seven- and eight-year-olds, four (16%) produced other than Level 4 drawings/comparisons, and it was difficult to make sense of their work (they seemingly had some idea in mind that had nothing to do with the task). One hundred percent of the nine-year-olds were scored at Level 4. Taken together, 89% of the seven- through nine-year-olds were scored at Level 4.

The six sticks task was useful in discerning the limits of four- and five-year-olds' symbolic representational abilities, but above the age of six, the task was too simple to be of interest. The older children considered the task to be silly, or thought that the Interviewer was asking for something far more complicated than she was.

Task 6b. Symbolizing Wheels.

Level 1. The child represents an object as such, and not a quantity of objects.

Example: the child draws a car or a truck and ignores the request to draw amounts of wheels.

Level 2. The child draws a quantity of objects.

Type A. The child draws many wheels, or an approximation of the ungrouped numerosity of the whole.

Type B. The child draws one or two sets of wheels.

Level 3. The child represents the numerosity of the whole. In the process of drawing, however, she transforms the objects into something else (e.g., "hamburgers," or "a rabbit and a dog"). Thus she abandons the idea of "groups of four."

Note: the shifting of ideas midstream indicates how fragile the quantitative idea is (a whole of sixteen, and within that whole, sub-collections or sets of four).

Level 4. The child represents the numerosity of the whole (sixteen objects) as well as sub-collections or groups of four. The sets are indicated in one or a combination of the following ways — color (a different color for each of the groups of four), spatial grouping (sets of four drawn on different areas of the paper), boundary lines (lines indicating the separation of the whole into groups of four), and labeling (numerals and/or written words to identify the groups).

After children had worked with the Tinkertoy wheels, they were asked to make a drawing of the wheels so that "someone who came along and looked at it" could tell that there were as many wheels as there were, and that four cars could be outfitted with them. Level 1 children drew an object as such, usually a car or a truck, and ignored the Interviewer's specific request for "a drawing of the wheels." They represented an object, one final product, rather than a quantity of objects. (See Appendix A for an example of this and other levels.) At Level 2, children did represent a quantity of objects, but they amounted to an ungrouped multiplicity of wheels (a vague approximation of the whole which was arrived at as soon as the paper was filled with wheels, thus leaving "no more space") or one or two sets of wheels (the second set being produced after a good deal of prodding from the Interviewer).

Level 3 represents a mixture of responses that emanate from children's loss of their original goal, or put differently, from their inability to carry an idea through from beginning to end. All of these children drew one set of four objects at the outset; but in the process of making their second or third set, they allowed extraneous ideas to enter into their heads which diverted them from the original task (e.g., "Hey, that looks like a rabbit," or "Those are hamburgers, I make them like that"). By the end they could not recall what it was that they had started out to make. It was as if their lack of a firm notion of "four wheels and four groups" allowed them to "stray from the task." In contrast, Level 4 children represented both the numerosity of the whole (sixteen objects) and the sub-collections (sets of four). They used one or a combination of the following graphic supports: spatial separation between sub-collections; boundary lines around groups of four; different colors for each of the sets of wheels; and labeling with words and numerals.

More than half of the four-year-olds were at Level 1 in this task, 27% were at Level 2, and only one child reached Level 4. The proportions were the reverse for the five-year-olds: only one child was at Level 1, 39% were at Level 2, and almost half (46%) reached Level 4. Among the six-year-olds, one-sixth were at Level 2, an equal proportion were at

Level 3, and two-thirds of the children performed at Level 4. Eighty-eight percent of the seven-year-olds, and 92% of the eight- and nine-year-olds were scored at Level 4. Considered together as collapsed age groups, one-third of the 24 four- and five-year-olds were categorized at Level 1, an equal number were at Level 2, a single child was at Level 3, and 29% were at Level 4. Among the 29 six- and seven-year-olds, 15% were at Level 2, 7% (two children) were at Level 3, and 79% were at the highest level. Of the 24 eight- and nine-year-olds, all but two children were categorized at the highest level (92%). These results have to be considered alongside the results of the gum task, and it is to those drawings that we now turn our attention.

Task 7b. Symbolizing Gum.

Level 1. The child represents an object as such, and not a quantity of objects. For example, the child draws one or two sticks of gum and vacillates between identifying the objects as "sticks" and "packs."

Level 2. The child draws a quantity of objects.

Type A. The child draws many sticks of gum, or an approximation of the ungrouped numerosity of the whole.

Type B. The child draws the sticks for one or two packs of gum.

Level 3. The child represents the numerosity of either the ungrouped whole (all of the sticks, but none of the packs), of the sets (all of the packs but none of the sticks), but not both.

Level 4. The child represents the numerosity of the whole as well as the groups. The child draws either twenty-three sticks clustered into groups of five, with three sticks "left over," or four packs and three sticks.

The levels for evaluating children's drawings in the Packs of Gum task are the same as those for Wheels and Cars discussed above. The reader should be reminded that fewer children (particularly four- to six-year-olds) completed this task than was the case for Task 6b. When the results of the two tasks are compared, it becomes evident that the four- through six-year-olds responded differently than the seven- through nine-year-olds.

Among the four-year-olds who tried the gum task (two-thirds of the sample), 25% were scored at Level 1 and 63% at Level 2. A single child reached Level 3. Among the five-year-olds (again two-thirds of the sample), the distribution was 10% at Level 1, 40% at Level 2, 30% at Level 3, and 20% at Level 4. Of the six-year-olds (83% of the sample), the proportions changed to 30% at Level 2, 30% at Level 3, and 40% at Level 4. The percentages for collapsed age groups were as follows. Of the four- and five-year-olds, one-sixth or 17% were at Level 1, 50% were at Level 2, 22% at Level 3, and 11% (two subjects) at Level 4. For the 26 six- and seven-year-olds, 15% were at Level 2, 19% at Level 3, and 65% at Level 4. Among the 24 eight- and nine-year-olds, all but a single subject were categorized at the highest level (96%).

Four points need to be made about these results. First, Task 7 was harder than Task 6 for reasons discussed earlier: there were remainders; the group size was five rather than four; and the child was asked at the outset to imagine actions completed in the past. Because it was awkward to ask the younger children to draw something they had not seen, they were in fact asked for a drawing of the amounts they had worked with.

Second, for the four- and five-year-olds, n = 13 was tantamount to "beaucoup," even though many of them had counted that many sticks of gum with apparent ease. N = 23 was too large for the youngest children and was manageable for some but not all six-year olds. Twenty-three was still a large amount for some of the younger seven's to draw, but by and large they were able to make a drawing of that many sticks and to represent the objects as grouped quantities.

Third, some of the older children were able to learn from the earlier wheels task and showed improvement in their level of performance on the gum task. However there is a marked decline (26%) in the proportion of five- and six-year-olds who reached Level 4. Task 7b was clearly much too difficult for the four-year-olds to manage, and it was quite hard for most of the five-year-olds as well. The distribution of six-year-olds' scores (30% at Level 2, 30% at Level 3, and 40% at Level 4) suggests that the task might be useful and worthy of replication among children aged six years and older.

Fourth, among the seven- through nine-year-olds, children's drawing levels on the gum and wheels tasks remained substantially the same. Two of the seven-year-olds, and one nine-year-old had more difficulty with Task 7b than Task 6b, but for the most part the results were stable. This forms quite a different picture than the one formed by the younger age group.

Task 9. Marbles.

The marble game was the last task that children were asked to do, and by that point many children "just wanted to play" and asked the Interviewer to do the score-keeping. Among these children, some were reluctant to try to invent a means of keeping score "without using numbers." But most of them were getting tired and simply wanted to play. The following is a list of methods employed by the children who did keep score, but they do not constitute "levels." It was interesting to note that no child who used an invented procedure had any difficulty coming up with some means of recording "zero."

Type 1. Tally marks (no necessity for writing anything down for zero).

Type 2. Alphabetic letters (i.e., A for 1, B for 2, C for 3, and so forth).

Type 3. Arbitrary "ideographs" (e.g., a "happy face" for 1, an "x" for 2, a slash-mark for 3, and so forth).

Type 4. A drawing of a pie, where an appropriate number of pieces were colored in for the number of marbles knocked out on each shot.

Developmental Levels in Conventional Representation

Tasks 6c, 7c and 8. Writing Numerals.

Level 1. The child lables individual objects with some kind of mark. For example, she makes a sequence of short lines, one mark underneath each object drawn.

Level 2. The child makes a mark approximating the shape of the number-squiggle. He often makes such remarks as, "A five, that's a backwards two," or "That's how I make a six," as he writes the squiggles.

Examples: for 3; for 5; for 6; for 7.

Level 3. The child makes the conventional mark (appropriate shape) for most single-digit numerals; two-digit numerals are often inverted.

Examples: "61" for 16; "21" for 12.

Level 4. The child writes the conventional marks for all numerals.

Among the children interviewed for this study, the ability to write number-squiggles begins at age four; by the time they are seven years old, children are able to write two-digit numerals with some facility. The levels reflect the following sequence of acquisition: (1) knowledge that each object represented on paper is identified with a specific number name and some kind of a discrete mark; (2) an awareness of the shapes used to represent number-names, and a growing facility with making those shapes on paper; (3) an ability to make the conventional marks for single digit numerals, but confusion over the order in which to write the two squiggles for numbers greater than nine; and (4) an ability to make the conventional marks for two- and three-digit numerals in their correct order.

One-third of the four-year-olds were at Level 1; they were the only children in the entire sample who could not write any numerals. Fifty-eight percent of the four-year-olds and 46% of the five-year-olds were in the process of learning how to make the specific squiggles. The remaining 54% of the five-year-olds were quite proficient in making all nine digits with only an occasional lapse and were scored at Level 3. None of the four- or five-year-olds knew how to make two-digit numerals without help. Fifty-eight percent of the six-year-olds were also at Level 3, and the remaining 42% were able to write two-digit numerals. Eighty-eight percent of the seven-year-olds, and 100% of the eight- and nine-year-olds were categorized at Level 4. When age groups were collapsed, the following percentages were found. Among the 25 four- and five-year-olds, 16% were categorized as Level 1, 52% as Level 2, and 32% as Level 3. None of these children approached Level 4. Of the 29 six- and seven-year-olds, no one was at Level 1 and only a single child was at Level 2. Twenty-eight percent of the subjects were at Level 3 and 69% at Level 4. One hundred percent of the eight- and nine-year-olds were categorized as Level 4.

It is noteworthy that children who are simultaneously acquiring the alphabetic writing system spontaneously remark on the similarity between an *S* and a *5*, a *B* and an *8*, a *P* and a *4*, or an *E* and *m* and *3*. These kinds of comparisons illustrate children's search for ways to make the elements of both the alphabetic and numerical representational systems sensible for them.

Among the five- and six-year-olds, "reversals" in writing two-digit numerals were the rule rather than the exception. For the "teen" numbers there is ample reason for this state of affairs: one hears the sound "six" before the root "teen," and our left-right order of recording spoken words (reading) supports this error. Interestingly enough among the pre-readers, the question "which side does the 1 go on" for 10 and 12, was raised as often as it was for the "teen" words.

Task 6d. Interpreting, or Assigning Meaning to Digits and Numerals.

Level 1. Number-squiggles are graphic marks that are linked to the objects on which they are found (F. Siegrist and A. Sinclair, research in progress).

Type A. Number-squiggles are "naming labels."

Example: 4 "says car" or 2 is "Channel 2."

Type B. Number-squiggles carry "functional messages."

Example: "they're for things you buy" or 0 (zero) is "for blast-off."

Type C. Number-squiggles have no direct relation to anything written or represented on the paper. The child might circle things because the Interviewer circled things (albeit their number-squiggles).

Level 2. Single-digit number-squiggles are generally recognized and called by their appropriate name (e.g., "that's the number six"). But rather than making quantitative correspondences between numerals and represented objects, the child seems to use some kind of a "matching schema" to make a link between squiggles and other things. These correspondences are for the most part non-quantitative, though quantitative notions are occasionally mixed in.

Type A. The child makes a correspondence between the colors used in writing squiggles and drawing objects.

Type B. The child makes a verbal (number-name) correspondence between a squiggle and some unrelated instance in which that name is known.

Example: 4 (written to show "how many wheels") elicits, "I know that because I'm four years old," or "I was four before I was five." The connection between the 4 and four wheels, drawn by the child just moments before, is not made.

Type C. The child makes a correspondence between one number-squiggle and any other number-squiggle written on the paper, as if to say "they're both numbers, and therefore they 'match.'"

Type D. The child makes a correspondence between identical number-squiggles. The correspondence is qualitative (identical mark) rather than quantitative (identical mark to signify same amounts).

Level 3. Number-squiggles, and particularly single-digit numerals, can stand for quantities of represented objects. But other ideas operate at the same time, resulting in confusion and inconsistency of responses. The notion that single- and two-digit numerals refer to specific amounts (cardinality) is one among several ideas that are not fully differentiated, one from the other.

Type A. Two-digit numerals cannot be "dissected" into their constituent digits. The number "disappears" when it is broken down into its written parts.

Type B. A whole two-digit numeral, as well as either written part, all refer to the same amount.

Type C. The objects drawn can be used to answer one question, but they cannot serve as a referent to answer the second question.

Example: The child draws a line around six objects for 16, but can consider only the remaining ten objects when asked about 16.

Type D. Number-squiggles as "ordinal labels" (that is, a sequence of marks identifying separate objects in a sequence of objects) is not differentiated from number-squiggles as signifying "cardinal values."

Example: the 6 in 16 means the sixth wheel; or the whole numeral 16 means the sixteenth wheel.

Type E. In the process of searching for meanings for the separate digits of a two-digit numeral, the "units of meaning" or "referents" change.

Example: the 6 in 16 refers to six wheels, but the 1 in 16 means one car (i.e., six of something and one of something else).

Type F. The operation of addition is applied to the digits making up a two-digit numeral.

Example: the 1 in 16 means one wheel, the 6 in 16 means six wheels, and the whole numeral means "one and six is seven."

Type G. The shape of the graphic mark is selected as the focus of meaning.

Example: the child makes a figurative correspondence between the shapes of the numerals (1 is "like a line" and 6 is "like a circle") and other things drawn on the paper. Alternatively, circles drawn around the number-squiggles result in products that "look like a wheel or a machine."

Type H. A numerical correspondence is made between one, but not both, of the written parts of a two-digit numeral and objects.

Example: the 1 in 16 means ten, but the 6 in 16 means nothing at all; or the 6 signifies six objects, but the 1 means nothing.

Level 4. Whole two-digit (and later in Task 8, three-digit) numerals stand for the totality of the objects represented. The individual digits are consistently transformed into numerals in their own right, and they are treated in one of two ways. In neither case does the child sense a necessary relation between the numerical parts (six objects and ten objects) and the numerical whole (sixteen objects) being represented.

Type A. 1 in 16 signifies one object and 6 in 16 six objects; that nine objects remain unaccounted for is of no concern.

Type B. 1 in 16 stands for sets of one, and 6 in 16 for sets of six objects.

Level 5. The individual digits making up a two-digit (and later in Task 8, three-digit) numeral stand for amounts that are determined by the place or position in which the digits occur. The mechanisms leading to this understanding of place value consist of a synthesis of three gradually constructed ideas:

(a) Notational rule - 1 in 16 stands for ten because it is written in the tens place.

(b) Numerical part-whole relations - 1 in 16 stands for ten because six and ten add up to sixteen.

(c) Multiplication - 1 in 16 stands for ten because 1 x 10 equals ten.

The final set of levels summarizes the sequence of children's ideas regarding the meaning of digits and numerals. At the first level, children think that number-squiggles are linked to specific objects (e.g., cars), to events associated with the squiggle or the number-name ("ten, nine, eight ... blast off!"), or carry some kind of a functional message "they're for things to buy"). One-third of the four-year-olds interviewed for this study were categorized at Level 1. Level 2 subjects try to find some kind of correspondence between the squiggles they have written and something else on their paper. These correspondences are, for the most part, qualitative or figurative in nature, and they are arrived at by means of a "matching schema" or "matching strategy." The correspondences include (a) same color (blue squiggles and blue wheels), (b) same number name (four wheels and four year old), same squiggle (the same numeral written somewhere else on the paper), and (d) some other number-squiggle. Forty-two percent of the four-year-olds and 31% of the five-year-olds responded with one of the above "matches."

The next level (one hesitates to even call it a level), or collection of ideas consists of a series of beliefs that compete with strictly quantitative notions. These beliefs serve to confuse the child and lead him to respond, now with appropriate (numerical), and now with inappropriate (non-numerical) answers. For these children the notion that "two-digit numerals refer to the cardinal value of a drawn collection" is one among several ideas that are not fully differentiated, one from the other. The non-numerical ideas are as follows: (A) if two-digit numerals are dissected into their constituent digits, the number disappears; (B) both the individual digits and the whole two-digit numeral refer to the same quantity; (C) one of the two digits can refer to the number of objects represented but the whole numeral cannot; (D) number-squiggles label separate objects in a collection, but they do not simultaneously signify cardinal value (i.e., they are ordinal labels for individual objects in a sequence of objects); (E) the two digits can have totally different referents; (G) one of the arithmetic operations is imposed upon the digits, or an idea such as "odd and even numbers" is put forth; and (H) a numerical correspondence is possible between one, but not both of the digits and the objects. Twenty-five percent of the four-year-olds, 62% of the five-year-olds, 36% of the six-year-olds, 31% of the seven-year-olds, and a single nine-year-old were categorized as responding in a Level 3 way.

At Level 4 individual digits, whether they stand alone or in combination as multi-digit numerals, are consistently treated in one of two ways: as if they were numerals in their own right; or as if they were meant to signify sets of up to nine objects each. What is faulty with this otherwise reasonable theory is that the quantities thus signified never equal the numerical whole (Type A), or only equal the whole when that number happens to be equally divisible by the number suggested by the digit (Type B). It is evident that in either case, the child does not sense a necessary relation between the numerical parts and the numerical whole being represented. Fifty-five percent of the six-year-olds, 56% of the seven-year-olds, 82% of the eight-year-olds, and 50% of the nine-year-olds were categorized in Level 4. None of the four- or five-year-olds were at Level 4.

As implied above, the critical difference between Level 4 and Level 5 subjects is that the latter understand the numerical part-whole relations implied in place value: the digits stand for quantities that together make up the numerical whole signified by the multi-digit numeral. They have gone beyond learning the notational rule that a digit written in the tens place (the second position to the left of the decimal point) stands for a decadal number. While none of the children interviewed for this study had any familiarity with powers of ten, some of them were able to apply their knowledge of multiplication to explain that "two times ten is twenty, two tens are twenty." Two out of sixteen seven-year-olds, two our ot eleven eight-year-olds (18%), and five out of twenve nine-year-olds (42%) achieved the highest level.

The percentages for collapsed age groups are as follows. Five out of 25 four- and five-year-olds (20%) were at Level 1, 36% at Level 2, and 44% at Level 3. None of these children reached Levels 4 or 5. Of the 27 six- and seven-year-olds, a single child was at Level 2, 33% were at Level 3, 55% were at Level 4, and 7% (two children) attained the highest level. For the oldest group, no children were at either of the two lowest levels, a single child was at Level 3, 65% were at Level 4, and 30% were at Level 5.

Level x Age Analyses

Tables 1-10 (see Appendix C) contain the results of the level x age analysis for each of the tasks. The number of children whose responses could be categorized according to the levels described above are given, first by one-year age groups (4 year olds, 5 year olds, ... 9 year olds), and then by two-year collapsed age groupings (4 and 5 year olds, 6 and 7 year olds, and 8 and 9 year olds). The Chi square (χ^2) statistic and Cramer's statistic ($v = \sqrt{\frac{\chi^2}{N\ (\min\ (r-1),\ (c-1)}}$), calculated for collapsed age groups, is given for each table (Cramer, 1946).

All of the results of the Chi square test for this group of data were significant at the .001 level. The measure of association, while difficult to interpret, was high for many of these data. Four of the five cognitive tasks yielded v's that were .500 or stronger; only one (Task 6a, grouping sixteen wheels into sets of four, summarized in Table 5) had a weaker measure of association ($v = .391$).

The analysis of the three tasks used to appraise children's symbolic representational abilities netted slightly weaker results. The age x level analysis of Task 5 (representing six sticks in different spatial arrangements) yielded a v of .499 (see Table 4), and Task 6b (symbolizing grouped wheels) a v of .474 (see Table 6).

The results of the conventional notational tasks were collapsed from the outset. Children's writing of numerals in Tasks 6c, 7c, and 8 were evaluated and scored as a whole (Table 9), as was their assigning meaning to the numerals in Tasks 6d, 7d, and 8 (Table 10). The measures of association for both of these sets of data were above .500. Table 4 summarizes the within task level x age analyses.

Task	Chi square (χ^2)	Level (p)	Cramer's (v)
Conservation of Elementary Number (Table 1)	54.34, df = 6	.001	.583
Establishing Equality among Unequal Collections (Table 2)	48.47, df = 6	.001	.550
Socks and Pairs (Table 3)	56.02, df = 4	.001	.619
Six Sticks (Table 4)	35.39, df = 6	.001	.499
Grouping Wheels in Action (Table 5)	24.51, df = 6	.001	.391
Symbolizing Wheels (Table 6)	34.55, df = 6	.001	.474
Grouping Gum in Action (Table 7)	52.74, df = 6	.001	.614
Symbolizing Gum (Table 8)	36.99, df = 6	.001	.522
Writing Numerals (Table 9)	66.67, df = 6	.001	.671
Assigning Meaning to Numerals (Table 10)	129.26, df = 8	.001	.587

Table 4. Summary of level x age analyses for all tasks

This concludes the description of developmental levels for the tasks used in this study. These results will form the basis for the next phase of data analysis in which the evidence is considered in light of the hypothesized relations, summarized in Figure 10. Before proceeding with that analysis, however, some additional findings with respect to children's interpretations of digits and numerals should be described. All of the children were categorized into unique levels for Tasks 6d, 7d and 8, and that categorization was based upon what appeared to be their dominant idea across several tasks. In part the collapsing of children's notions into a single summary level was done in order to facilitate the formal analysis. However this "forcing" of the data into exclusive categories masks other findings which may, in fact, more accurately portray children's development with respect to knowledge regarding what digits and numerals can mean.

When the results for each child are considered in their entirety, another picture emerges. Individual children differed in the number of ideas they expressed in the course of engaging in several tasks. Two patterns emerged as a result of considering these differences. The first was that children in the two extreme age groups, that is, four-year-olds and nine-year-olds, tended to express a single idea with respect to the relationship between the notational marks and objects drawn. Eighty-three of the four-year-olds, and 67% of the nine-year-olds, gave a single interpretation. In contrast, only 45% of the five- through eight-year-olds stayed with a single idea. They were more inclined to hold two, three, and sometimes four different notions about that relationship. Furthermore these children were less willing to make categorical statements about which among their ideas was a better one.

The second pattern is concerned with the sequence of ideas used by children, whether singly or in combination, in the various tasks. The four- and five-year-olds' ideas were confined to Levels 1, 2 and 3: the bulk of the four-year-olds expressed Level 1 and Level 2 ideas, while the vast majority of five-year-olds used Level 2 and Level 3 ideas. The six- and seven-year-olds were focused on Level 3 and Level 4 notions: among the six-year-olds Level 3 ideas predominated, but among the seven-year-olds, Level 3 and Level 4 notions appeared in equal measure. Six of the sixteen seven-year-olds at least mentioned some aspect of place value (tens and ones), but only two of those six thought that "one ten" or "tens" or "two tens" bore a quantitative relationship to the number of objects drawn.

With the exception of a single child, the eight- and nine-year-olds used exclusively Level 3, Level 4, and Level 5 ideas. The instances in which a Level 3 notion was expressed dropped considerably with age: five out of eleven eight-year-olds (45%) used a Level 3 idea at least once, while only two out of twelve nine-year-olds (17%) did the same thing. Nine of the eleven eight-year-olds (82%), and seven of the twelve nine-year-olds (58%), expressed Level 4 ideas. Only two of the eight-year-olds (18%) thought that the Level 5 idea was superior, while five of the nine-year-olds (42%) expressed certainty about the superiority of the Level 5 interpretation.

CHAPTER VI

RESULTS:

RELATIONSHIPS AMONG LEVELS

The quantitative results bearing upon the seven exploratory hypotheses are reported in this chapter. Before considering these results, it would be well to reiterate some of the assumptions underlying the empirical search. First, it was assumed that children's learning and reconstruction of the conventional notational system would utilize distinct cognitive abilities, some of which were examined here. Second it was presumed that individual cognition is the source of symbolic representation, while culture is the repository of written systems: symbolization springs largely from within, and notation is in good measure transmitted from without. Finally, it was expected that the distance between conceptual development and symbolic representation would be closer than the distance between either of them and conventional representation.

The exploratory hypotheses reflecting these ideas fell into three categories: relationships within the domains of cognition, symbolic representation, and conventional notation; relationships between levels in the three domains; and questions regarding the strength of between-domain relationships. Tasks or analysis procedures were specified for each hypothesis/question (Figure 10, Chapter IV). This chapter reports the supporting and discounting statistical evidence for each hypothesis/question. The main tools for the analysis were contingency tables in which the levels reported in Chapter V constituted column and row variables. All of those tables can be found in Appendix D and Appendix E.

Relationships among Levels Within Domains

Hypothesis 1. There are levels (a) in the structuring of number, and (b) in the structuring of objects (whole collections) into groups (subsets or numerical parts).

Number concepts and numerical grouping abilities were two cognitive achievements thought to underlie place value understanding. Levels had been identified for each of the tasks involving conceptual development in number (Task 1, Conservation and Task 2, Establishing Equality) and numerical grouping (Task 4, Socks and Pairs; Task 6a, Grouping Wheels; and Task 7a, Grouping Gum). Children's levels in each one of these five tasks were tested, one against the other (see Tables 1D-10D, Appendix D). With the exception of one pair (levels in Task 6a x levels in Task 7a, Table 10D) the results of the Chi square tests for independence were significant at the .001 level. The measure of association was high (.500 or better) for all pairs except those involving Task 6a. These results are summarized in Table 5 on the following page.

The results suggest that the number concept tasks and grouping tasks involve abilities that develop simultaneously: insofar as these tasks, and the qualitative and quantitative analyses of them can determine, there are developmental levels in the cognitive domain. However the data do not allow us to extend the analysis to a more detailed explication of relationships between specific abilities associated with the gradual construction of number on the one hand, and the grouping of objects into subsets on the other. The extent to which they require the same or different capacities cannot be directly inferred. In part this limitation is a consequence of the fact that one of the conceptual tasks

(Task 3, Anticipating Equality between Unequal Collections without Counting) had to be dropped from the analysis. The loss of those data not only leaves us with an incomplete picture of conceptual development, particularly among children of seven years of age and older; it also precludes us from scrutinizing the data for finer-grained relationships.

Tasks	Chi square (χ^2)	Level (p)	Cramer's (v)
Task 1 x Task 2 (Table 1D)	88.02, df = 9	.001	.625
Task 1 x Task 4 (Table 2D)	65.36, df = 6	.001	.669
Task 1 x Tasl 6a (Table 3D)	49.16, df = 9	.001	.487
Task 1 x Task 7a (Table 4D)	61.83, df = 9	.001	.543
Task 2 x Task 4 (Table 5D)	77.50, df = 6	.001	.729
Task 2 x Task 6a (Table 6D)	46.96, df = 9	.001	.442
Task 2 x Task 7a (Table 7D)	52.69, df = 9	.001	.501
Task 4 x Task 6a (Table 8D)	32.67, df = 6	.001	.473
Task 4 x Task 7a (Table 9D)	51.16, df = 6	.001	.613
Task 6a x Task 7a (Table 10D)	25.60, df = 9	.005	.349

Table 5. Summary of relationships among levels in tasks designed to tap cognitive development

Hypothesis 2. There are levels in representing numerical quantities symbolically.

Tables 11D, 12D and 13D (Appendix D) analyze the relationships between children's levels on each pair of tasks calling for symbolic representation. Task 5 asked for drawings of Six Sticks arranged in three different ways; Task 6a was concerned with Wheels (number of objects = 16) and Cars (group size = 4); and Task 7a requested drawings of Gum (n = 13 or 23) and Packs of Gum (group size = 5). The results of the Chi square test for these data were significant at the .001 level. However the measure of the strength of association for pairs in which one member was Task 5 (Tables 11D and 12D) was less than v = .500. These results are summarized in Table 6.

Tasks	Chi square (χ^2)	Level (p)	Cramer's (v)
Task 5 x Task 6b (Table 11D)	29.57, df = 9	.001	.409
Task 5 x Task 7b (Table 12D)	36.47, df = 9	.001	.446
Task 6b x Task 7b (Table 13D)	62.92, df = 9	.001	.551

Table 6. Summary of relationships among levels in tasks requiring symbolic representation

The level x level analysis for each pair of symbolic representation tasks suggests that there may be common developmental threads in children's drawings of numerical quantities. But the results caution against trying to specify what those threads are. In addition it should be noted that the results of the between-task analyses are not as strong as they were for the cognitive (conceptual and grouping) aspect discussed above, or for the conventional representation domain reported below.

It should be emphasized that this analysis focused on children's drawings of numerical quantities, and deliberately ignored other features of children's symbolizing abilities that could be of interest for developmental studies of children's drawings. Children were asked to make drawings in order to find out (a) if they could represent numerical groups and subgroups, even if they did not know how to write digits;

(b) what kinds of ideas they might bring to bear on their drawn (symbolic) and written (conventional) representations; and (c) if there were any systematic differences in children's ideas that could be attributed to development rather than to specific learning. The drawings, as they were asked for in this study, were a particularly useful vehicle for eliciting the ideas that children do have that undoubtedly contribute to their problems in understanding place value.

Hypothesis 3. There are levels in acquiring the notational system (conventional representation). There are levels in writing number-squiggles; and there are levels in interpreting or assigning meaning to them. Embedded in this sequence are levels in grasping the place value property of the conventional notational system.

Children's knowledge of conventional notation was divided into two parts: their ability to write numerals; and their interpretation of the relationship between written numerals and symbolized objects. For the sake of having as complete a record as possible of each child's abilities, the results of the three tasks requiring conventional notation were collapsed from the outset. Therefore only one relationship remained to be tested, and that was their level in writing numerals against their level in assigning meaning to numerals. The results of that strong relationship are given in Table 14D in Appendix D ($\chi^2 = 115.1$, df = 12; $p < .001$; v = .751).

To the extent that the three notational tasks allowed children to expose their knowledge, and within the limits of the qualitative and quantitative analyses performed here, the results suggest that there are developmental levels in this domain. But the caveats regarding levels in Tasks 6d, 7d and 8 (Assigning Meaning to Digits and Numerals), spelled out in Chapter V, should be kept in mind as these results are evaluated.

Relationships among Levels Between Domains

Hypothesis 4. There is a positive relationship between levels in the cognitive tasks and levels in the tasks requiring symbolic representation.

The relationships between children's levels in the cognitive tasks and in the symbolization tasks are given in Tables 1E through 15E (Appendix E). The Chi square results for these data were significant at the .001 level. The strength of the association between variables ranged from v = .391 (Table 10E) to v = .735 (Table 9E). Although these results are difficult to interpret, it can be noted that all of the tables that yielded v's below .500 involved either Task 6a (the weakest of the within -task results as well as within-domain cognitive results) or Task 5 (the weakest of the within-task and within-domain symbolization tasks). The results of the between-domain analysis of levels in tasks designed to tap cognitive abilities, and levels in tasks requiring symbolic representation, are summarized in Table 7 which can be found on the following page.

Because this study was exploratory, tasks were designed to uncover a range of abilities that were thought to influence understanding: the tasks were not comparable, one to the other. The problem of pooling the results of different tasks into a single measure is a difficulty that haunts us as this stage of the analysis, for in order to talk about the strength of relationships between domains, some mechanism for evaluating that strength must be found. One way of beginning to assess between-

Tasks	Chi square (χ^2)	Level (p)	Cramer's (v)
Task 1 x Task 5 (Table 1E)	49.67, df = 9	.001	.486
Task 1 x Task 6b (Table 2E)	69.48, df = 9	.001	.571
Task 1 x Task 7b (Table 3E)	74.46, df = 9	.001	.604
Task 2 x Task 5 (Table 4E)	39.14, df = 9	.001	.429
Task 2 x Task 6b (Table 5E)	64.90, df = 9	.001	.527
Task 2 x Task 7b (Table 6E)	54.96, df = 9	.001	.519
Task 4 x Task 5 (Table 7E)	36.16, df = 6	.001	.508
Task 4 x Task 6b (Table 8E)	47.99, df = 6	.001	.566
Task 4 x Task 7b (Table 9E)	72.39, df = 6	.001	.735
Task 6a x Task 5 (Table 10E)	30.76, df = 9	.001	.391
Task 6a x Task 6b (Table 11E)	85.53, df = 9	.001	.605
Task 6a x Task 7b (Table 12E)	36.69, df = 9	.001	.424
Task 7a x Task 5 (Table 13E)	49.38, df = 9	.001	.511
Task 7a x Task 6b (Table 14E)	47.34, df = 9	.001	.475
Task 7a x Task 7b (Table 15E)	55.54, df = 9	.001	.552

Table 7. Summary of relationships between levels in cognitive and symbolic representation tasks

domain strength is to calculate an average for the individual measures of association. For Table 7, the average of the fifteen measures is v = .525 (range from v = .391 to .735, as pointed out above). Admittedly this is an impoverished approximation of relative strength between broad sets of relationships. But given the theoretical and empirical limitations of this study, it is at least one way of describing the results in quantitative terms. We will return to the discussion of relative strength between domains in the treatment of Hypothesis 7 below.

Hypothesis 5. There is no consistent relationship between levels in the cognitive tasks and levels in the tasks eliciting knowledge of conventional representation.

The relationships between children's levels in the cognitive tasks and their level in writing numerals (Tasks 6c, 7c and 8) are described in Tables 16E-20E (Appendix E) and summarized in Table 8 below.

Tasks	Chi square (χ^2)	Level (p)	Cramer's (v)
Task 1 x Tasks 6c, 7c and 8 (Table 16E)	81.84, df = 9	.001	.611
Task 2 x Tasks 6c, 7c and 8 (Table 17E)	105.73, df = 9	.001	.672
Task 4 x Tasks 6c, 7c and 8 (Table 18E)	82.65, df = 6	.001	.747
TAsk 6a x Tasks 6c, 7c and 8 (Table 19E)	55.48, df = 9	.001	.487
Task 7a x Tasks 6c, 7c and 8 (Table 20E)	68.25, df = 9	.001	.570

Table 8. Summary of relationships between levels in cognitive tasks and tasks requiring that numerals be written

The relationships between children's levels in the cognitive tasks and their level in the second aspect of conventional notation, assigning meaning to numerals (Tasks 6d, 7d and 8) are described in Tables 21E-25E in Appendix E and are summarized in Table 9 on the next page.

The Chi square results for all of these ten relationships were significant at the .001 level. The measure of the strength of association for seven of the ten between-domain tables was above v = .500. For the remaining three tables, the measure dropped below that point. Again

Tasks	Chi square (χ^2)	Level (p)	Cramer's (v)
Task 5 x Tasks 6d, 7d and 8 (Table 29E)	40.98, df = 12	.001	.448
Task 6b x Tasks 6d, 7d and 8 (Table 30E)	67.20, df = 12	.001	.554
Task 7b x Tasks 6d, 7d and 8 (Table 31E)	117.28, df = 12	.001	.776

Table 11. Summary of relationships between levels in tasks calling for symbolic representation and tasks assessing knowledge of conventional notation (interpreting digits and numerals)

Strength of Relationships Between Domains

Hypothesis 7. There is a stronger relationship between levels in the areas of cognition and symbolic representation than there is between levels in symbolic representation and conventional notation.

Hypothesis 7 is a straightforward question regarding the relative strength of between-domain relationships. It suggests that the statistical relationship between levels in the domains of cognition and symbolic representation should be stronger than the relationship between levels in the two domains of representation, that involving individually motivated symbolization, and culturally given notation. The evidence for this hypothesis lies in the results of the foregoing qualitative and quantitative analyses. It should be noted that each phase of data analysis, as it is presented here, takes us further and further from individual children's observable behavior.

Tables 10 and 11 summarize the relationships between levels, described in Tables 26E-28E and 29E-31E in Appendix E. These tables are concerned with making drawings of quantities against levels in writing digits on the one hand, and levels in assigning meaning to digits and numerals on the other. The Chi square results for all of these data are once again significant at the .001 level. The measure of association was above v = .500 for pairs in which Task 5 (the weakest of the symbolic representation tasks) was not involved. When the three measures of association in Table 10 are combined, the average is v = .601. For Table 11 the average strength of association between levels is v = .593. Overall, the average strength for all pairs in the two domains is v = .597. This is almost the same as the average for the cognitive and conventional domains discussed above (Hypothesis 5).

Tasks	Chi square (χ^2)	Level (p)	Cramer's (v)
Task 5 x Tasks 6c, 7c and 8 (Table 26E)	45.13, df = 9	.001	.467
Task 6b x Tasks 6c, 7c and 8 (Table 27E)	83.54, df = 9	.001	.598
Task 7b x Tasks 6c, 7c and 8 (Table 28E)	109.10, df = 9	.001	.737

Table 10. Summary of relationships between levels in symbolic representation tasks and conventional notation tasks (writing numerals)

Tasks	Chi square (χ^2)	Level (p)	Cramer's (v)
Task 5 x Tasks 6d, 7d and 8 (Table 29E)	40.98, df = 12	.001	.448
Task 6b x Tasks 6d, 7d and 8 (Table 30E)	67.20, df = 12	.001	.554
Task 7b x Tasks 6d, 7d and 8 (Table 31E)	117.28, df = 12	.001	.776

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Given the kinds of analysis done thus far, the most direct means of assessing the relative strength of between-domain relationships was to use the measures of association (Cramer's v) which had been calculated for each pair of tasks in the between-domain analyses (Appendix E). The procedure was simple. The v's for each pair of cross-domain tables were placed in one of three categories (cognitive/symbolic, symbolic/conventional, and cognitive/conventional), and an arithmetic average was found for the respective categories. The procedure is reproduced in Table 12 on the following page. The results are as follows: the arithmetic average of all of the measures of strength of association between levels in tasks tapping cognition and tasks using symbolic representation was $v = .525$ (range from $v = .391$ to $v = .735$); the average of the measures for tasks using symbolic representation and tasks calling for conventional notation was $v = .596$ (range from $v = .448$ to $v = .776$); and to complete the picture, the average for tasks designated as uncovering cognitive abilities and tasks requiring knowledge of notation was $v = .599$ (range from $v = .437$ to $v = .797$). On the basis of these calculations, Hypothesis 7 would have to be rejected, for $v = .596$ is clearly greater than $v = .525$.

However, we should be cautious about these results for several reasons. In the first place, an arithmetic average is no more or less than a simple average. As such, it is an impoverished descriptor of a complicated set of relationships. In the second place, Cramer's v is in itself a difficult statistic to interpret. This makes an average of all of the v's even more difficult to interpret. But there are other reasons as well. The strength of relationships involving conventional notation throughout the analysis may have been influenced by the fact that the

Strength of Association between Cognitive and Symbolic Tasks		Strength of Association between Symbolic and Conventional Tasks		Strength of Association between Cognitive and Conventional Tasks	
Tasks	Cramer's v	Tasks	Cramer's v	Tasks	Cramer's v
1 x 5	.486	5 x 6c, 7c, 8	.467	1 x 6c, 7c, 8	.611
1 x 6b	.571	5 x 6d, 7d, 8	.448	1 x 6d, 7d, 8	.566
1 x 7b	.604	6b x 6c, 7c, 8	.598	2 x 6c, 7c, 8	.672
2 x 5	.429	6b x 6d, 7d, 8	.554	2 x 6d, 7d, 8	.629
2 x 6b	.527	7b x 6c, 7c, 8	.737	4 x 6c, 7c, 8	.747
2 x 7b	.519	7b x 6d, 7d, 8	.776	4 x 6d, 7d, 8	.797
4 x 5	.508			6a x 6c, 7c, 8	.487
4 x 6b	.566			6a x 6d, 7d, 8	.437
4 x 7b	.735			7a x 6c, 7c. 8	.570
6a x 5	.391			7a x 6d, 7d, 8	.473
6a x 6b	.605				
6a x 7b	.424				
7a x 5	.511				
7a x 6b	.475				
7a x 7b	.522				
	Average v = .525		Average v = .596		Average v = .599

Table 12. Strength of the statistical relationships between domains: cognitive and symbolic, symbolic and conventional, and cognitive and conventional

results of three tasks (Tasks 6, 7 and 8) were collapsed from the outset. An important consequence of this initial pooling of data (which had been done for the sake of having as complete a picture of each child's ideas as possible) was to hide variations in individual children's knowledge in a way that was not true for either the cognitive or symbolic domains. In a word, the procedures used in categorizing children into levels were not precisely the same for all domains. In fact, one of the more interesting findings concerned age-related variations in the number of different ideas that children held, all at the same time, in responding to the second aspect of conventional representation. Seven- and eight-year-olds stood out as having a greater number of juxtaposed ideas about the relationship between digits, numerals, and drawn objects than either the four- through six-year-olds or the nine-year-olds. This finding will be further elaborated in the next chapter.

In addition it should be said that the levels for the individual tasks are neither a priori categories nor products of cross-task analyses. They describe a sequence of construction as it was reflected in children's behavior in the context of engaging in specific activities. The patterns of performance described in the contingency tables reflect the actual number of children who could be categorized by using the levels defined in Chapter V. It would be difficult to extend the analysis beyond what was done here.

CHAPTER VII

DISCUSSION AND CONCLUSIONS

This exploration into children's understanding of the place value property of our numeration system used research methods associated with the Genevan psychogenetic, and more standard hypothesis-testing experimental approaches. The results of this study reflect some of the strengths and some of the weaknesses of each of these methods of investigation.

In general, the major strength of the looser exploratory techniques allowed by the Genevan school lies in the extent to which the researcher capitalizes on the ideas expressed by children and follows their train of thought, rather than being narrowly constrained by a set of specific procedures designed to test well articulated hypotheses. The major weakness is that such techniques often yield complicated data, and as a consequence, the procedures used to describe the results are difficult to specify in advance. Data analysis proceeds with the same attitude of cautious probing that characterizes the initial questioning of children.

The major strength of more stringent and constrained experimental procedures is that specified variables, hypothesized to influence an outcome in particular ways, are isolated for scrutiny at the outset. The procedures by which the data have been collected and the results obtained are more amenable to precise replication and verification. The major weakness is that new possibilities that flow from children's behavior are foreclosed from investigation, at least at the moment of their occurrence.

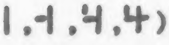
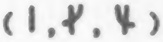
When psychological perspectives and research methodologies that are as different as the ones represented in this study are combined, a number of difficulties arise at each phase of the research. At the outset, hypotheses that were not formulated as testable statements had to be altered so that they could be treated as if they were assertions that lent themselves to confirmation or rejection. In point of fact, the hypotheses set forth in Chapter III were intended to serve as heuristic premises that would give direction to several exploratory and descriptive lines of study. They had a methodological we well as substantive aspect. Furthermore the hypotheses were not all of the same kind: they varied in the extent to which they reflected new as opposed to previously investigated problems; in the degree to which they dealt with general rather than specific relationships; in the extent to which they made use of widely replicated results rather than results gleaned from pilot research; and in their relative susceptibility to one or another kind of data analysis procedure. In addition the first group of hypotheses (numbers 1, 2 and 3) posited development in three conceptually distinct domains and had a different status than the ensuing group of three hypotheses (numbers 4, 5 and 6) that suggested more specific relationships between developmental lines among the domains, or the last hypothesis (number 7) that dealt with the strength of a statistical relationship among two between-domain pairs. In spite of all these differences, the data bearing on each of the hypotheses were subjected to uniform quantitative analyses.

The results of this study are, from both the Genevan and experimental standpoints, limited but yet of some interest. At the positive end of the continuum, the complexity of children's knowledge-building regarding an important cultural tool became evident far more quickly than would have otherwise been the case. At the negative end, the analysis had to be more global or general than one might have wished, and yielded statistical results that can only be regarded with hesitancy and caution as to what can, in fact, be concluded from them. This is not to say that no interesting patterns emerged from this study, for when the results for each child are considered in their entirety, developmental patterns do emerge in terms of the order in which ideas are constructed.

At the outset of this study, the basic action underlying place value was thought to lie in the ability to go back and forth between units and numerical groups, that is, to group units into multiples of ten and to partition higher order groups of ten into smaller groups or units. Symbolic representation was a mechanism by which children could show what they understood of the grouping action in graphic form, and served as a vehicle for exposing their ideas regarding the relationships between symbolized objects and quantities represented in conventional notational form. Children's knowledge of place value was investigated in relation to their interpretations of that relationship. Let us turn to the results of the interviews with children.

There is a fair distance between children's performance in the number concept and grouping tasks, and the ideas they expressed about the relationship of number-squiggles to symbolized quantities, especially at the older half of the age spectrum. One reason for the large gap can be attributed to a methodological dilemma faced in the designing of this study. Because baseline developmental results had not been established for most of the tasks, it was decided that all of the tasks should be given to all of the children in the sample. Therefore, caution had to be exercised regarding how difficult the tasks could be. It was feared that excessively hard conceptual and grouping tasks would inhibit the enthusiasm of the younger subjects, and by the same token, it was feared that drawing tasks that were excessively easy would alienate the older subjects. In a word, the same methodological dilemma influenced the complexity of the cognitive and the symbolic tasks, but in opposite directions. The adopted course of action was to keep the conceptual tasks quite simple and to limit the grouping tasks to small amounts (small wholes and small subgroups). The solution was only partially successful.

Another reason for the apparent distance between the cognitive results and the findings concerning conventional representation lies in the overly facile relationships that was presumed to exist between the two domains. Child-learners appear to have a number of theories, not directly linked to the cognitive capacities that were examined, that intrude upon any direct understanding of the elements or properties of the numeration system. These theories, or relationships, or connections between bits of knowledge, were articulated by different children in different ways.

Let us review the hypotheses that children brought to bear on number-squiggles as they worked through the many tasks. (1) They are marks that can be "read" and that might convey information, either about the objects on which they appear, or about some action or function that is associated with the object. (2) They are objects to write or draw that are similar in some figurative or qualitative way to other graphic marks that children know how to make. In fact, children who were in the beginning phases of attempting to write digits used their knowledge and skill, acquired in the context of learning to draw, to produce their shapes: for example, a "6" is a circle with a curved line on top (o ,∂); a "4" is a series of line segments (); it is also a straight line on top of which a curved line is drawn ().

Children who are learning how to write letters and digits spontaneously remark on the similarity between an "S" and a "5," a "B" and an "8," a "P" and a "4," and an "E" or "m" and a "3." Just as often a same-system comparison is made in approaching the task of writing digits: "a five is like a backward two" is one example. Finally, among pre-readers the question, "Which side does the 1 go on?" in regard to writing the squiggles for "ten" and "twelve" was asked as often as it was for the "teen" numbers. For kindergarteners and first-graders, "reversals" in writing two-digit numerals was the rule rather than the exception, particularly for the "teen" numbers in which the "six" is heard before the root, "teen." Our left-right order of recording spoken language (for spelling as well as for reading) quite clearly supports the error.

The use of previously acquired knowledge from art, the spontaneous use of untaught comparisons, and the tendency to "overregularize" the left-right sound-notation correspondence, all illustrate young children's search for ways to make the arbitrary elements of both the alphabetic and numerical sign systems sensible for them. The errors demonstrate that they are thinking.

(3) Another collection of ideas that predominate among five-year-olds, but that are found among older children as well, can be characterized in negative terms as confusion and inconsistency with regard to the quantitative referents for digits and numerals. The children who were categorized as belonging in this level did not seem to sense a need for consistency in their interpretations. A positive characterization of these children's thinking is a bit more difficult to make.

There is no question in these children's minds that number-squiggles can, and for the most part do have something to do with the symbolized quantities. Furthermore the ideas they put forth are in some way related to number: to an aspect of ordinal number (but as labels rather than as full blown ordinals); to an arithmetic operation that is being studied in school (addition for first graders, multiplication for third graders, and division for fourth graders); to odd and even numbers; and so forth. These children acted as if they were searching their repertoire of number-related ideas, acquired in the context of math activities in school, and applying one of the remembered actions to the novel out-of-school experimental tasks in something of a trial-and-error way.

The appearance of this kind of strategy among five- and six-year-old children is not surprising. Their confidence in dealing with quantities larger than ten or twelve is not high, and their exposure to written number is limited. But when it appears in the behavior of older children, there is cause for some concern. The study of numbers involves logico-mathematical knowledge, and mathematics as a discipline endeavors to describe relationships in precise and coherent ways. When older children do not discard one of their clashing ideas as "not applicable" or "a false lead," or when they seem reluctant to make a choice from among competing ideas, we ought to pay attention. Why are these children not making more of the clash? Is it because they are not thinking through a given relationship? Is it because their understanding of, and confidence in a given relationship is too shaky to reconstruct it for use in a novel situation? Is it because no single relationship among the possible relationships seems to be a more promising possibility than any other? It would seem that mathematical knowledge, to be useful to an individual, must consist of relationships that are amenable to personal reconstruction. In the absence of further information regarding the older children's knowledge of such operations as division, or such properties of numbers as odd and even, it is difficult to tell whether the inconsistencies in their interpretations are primarily a function of not understanding place value, not understanding the operation or property raised as a possible explanation, not believing in the necessary coherence of notational-numerical relationships, or some other factor.

(4) The theory that the digits used to write two- or three-digit numerals represent either units, or sets of the size indicated by the digits, was held by more than half of the six- and seven-year-olds, an even larger proportion of the eight-year-olds (80%), and half of the nine-year-olds interviewed in this study. The powerful hold that these ideas have on the thinking of six- through nine-year-olds carries us some distance in understanding children's resistance to place value instruction. In the minds of these children, numerals such as 23 or 105 represent whole amounts, but the digits in 23 or 105 represent one, two, three, or five objects, or sets of one, two, three, or five objects each. Children subscribing to the latter view first divided a whole symbolized collection (e.g., twenty-three objects) into sets of three, and then went back over the entire collection and marked the objects as sets of two. Children who believe in either the "units" or "sets" ideas probably regard the lesson that the digits refer to two and three of something else (two sets of ten and three units) with skepticism and disbelief. The idea that the digits have values that together correspond to the numerical whole does not arise as an issue.

Several remarks may be made in regard to the tenacity of these ideas. First, they are ideas that children systematically apply in task after task. They are in this sense general, powerful, and well-learned notions. It is likely that they have undergirded much of the child's understanding of number thus far, and may be good indicators of the extent to which the child has constructed "units composed of units" (Steffe _et_ _al_., 1981) for one aspect of number. The resistance to restructuring these ideas for place value understanding becomes more intelligible from this point of view, for _other_ structures have to be coordinated with this idea. Among them the most important may be the second order multiplicative operation of "one, two, ... ten, eleven, ... n groups of ten" (i.e., _13_, _23_, ... _103_, _113_, ... n).

Second, the curricular time table for instructing children in place value is determined by the role it plays in explaining the regrouping procedure in multi-digit addition and subtraction. However it is quite possible to perform addition and subtraction problems involving regrouping without regrouping at all. Children can rely on their more solid knowledge of small numbers (less than ten) to solve these problems, in spite of laborious or clever instruction. In point of fact, when children are asked to describe their procedures as they are working through problems involving regrouping, they speak in terms of "carry the one" in addition and "borrow the one" in subtraction, and rarely, if ever, refer to the carried and borrowed digits as ten. "Tens" and "ones" are the names of the second and first columns, from right-to-left, and what is written in the columns are numerals rather than digits.

Third, in other areas of knowledge-building, the "what for" question ("what is it good for?") and "how to" issue ("how do you use it?") generally precede any attempt at description or explanation of the relationships involved. This is curiously untrue in the case of place value instruction: the presumption seems to run in the opposite direction. Children should understand the rationale for what they are doing in order to grasp the activity itself. The "what for" and "how to" that presumably motivate curiosity about why something works -- about why "carrying" and "borrowing" work, and why numerals are "spelled" in the way that they are -- are lacking.

(5) The criterion for categorizing children as having understood place value was defined as their ability to show the correspondence between digits and numerals, and symbolically represented objects. Accordingly a small proportion of the seven- and eight-year-olds, and a larger percentage of the nine-year-olds (just under half) interviewed for this study grasped place value. They showed the digits 6 and 1 in the numeral 16 as standing for six objects and ten objects respectively, and the numeral as standing for the entire collection. They were able to indicate the quantitative referents for at least two "teen" numbers, two numbers in the twenties, and two numbers in the hundreds.

These children had apparently coordinated many different ideas into a system of relationships that enabled them to understand place value. The ideas that the children had synthesized included the following: (a) the notational principle that written location matters; (b) the correspondence between written position (columns), column names (ones vs. tens vs. hundreds) and digits (two vs. two-tens vs. two-hundreds); (c) the idea that digits can be freed from the numerals in which they occur, and that they signify numerical quantities that are different from the quantity signified by the numeral (the exception is numbers that end in a zero); (d) the sum of the quantities represented by the digits is the same as the quantity represented by the numeral. What differentiated these children from those categorized at lower levels was the extent to which all of the ideas and specific learnings were synthesized into a system of relationships.

Among the seven- and eight-year-olds, several children responded to the more school-like Task 8 with Level 5, place value ideas, in spite of the fact that they were quite at ease with their Level 3 or Level 4 responses to the Wheels (Task 6) and Gum (Task 7) problems. Those who did not feel the need to correct their earlier interpretations were categorized at the lower level; those who did note the inconsistency, and felt the need to correct the erroneous responses, were categorized at the higher level. The desire to make a choice between clashing ideas was expressed by the nine-year-olds alone.

One is left wondering why the seven- and eight-year-olds whose clashing notions were mentioned above, displayed thinking that seemed to be so compartmentalized. How this compartmentalization (or juxtaposition of ideas) eventually yields to some higher-level synthesis, as it did for some nine-year-olds, is a matter of great curiosity. It is possible that children, as well as adults, have to experience the same ideas or lessons in a variety of specific contexts before it occurs to them that there might be some relationship (similarity/difference) that has been overlooked. The construction of more adequate ideas, or better theories concerning number notation may be born of opportunities to think about the possible connections between hitherto compartmentalized, specific learnings.

But it is equally possible that better theories are the product of relationships constructed in areas other than the one in which it shows up. For example, it could have been the hard thought put into figuring out division that accounted for the better performance of the nine-year-olds in the tasks requested in this study. Division is the reciprocal of multiplication, and multiplication underlies the mathematical idea of powers of ten. Perhaps the enriched understanding of the place value property was a function of coming to grips with the more sophisticated, second order operations.

The results of this exploratory and descriptive study suggest that children's understanding of the place value property of the numeration system, rather than being constructed all at one time and in relative isolation from other learning, is built in phases, over a long period of time, in conjunction with other kinds of knowledge. The most fruitful approach to describing the process may lie in the notion of theory-building about the notational system as a whole, about number concepts, operations, and their interrelations.

It may be that some form of ordinal analysis such as the one suggested in Figure 11 might provide a better description of how development in the area of knowledge-building about numerical representation proceeds. In Figure 11, the solid lines indicate that a large number of children of a given age hold ideas described by the respective levels. The broken lines show that some, but not many, children consider ideas described by other levels. Such analyses are currently being explored by developmental psychologists such as Siegrist, Sinclair and Sinclair (Geneva) and Phelps, Wolf and Gardner (Cambridge).

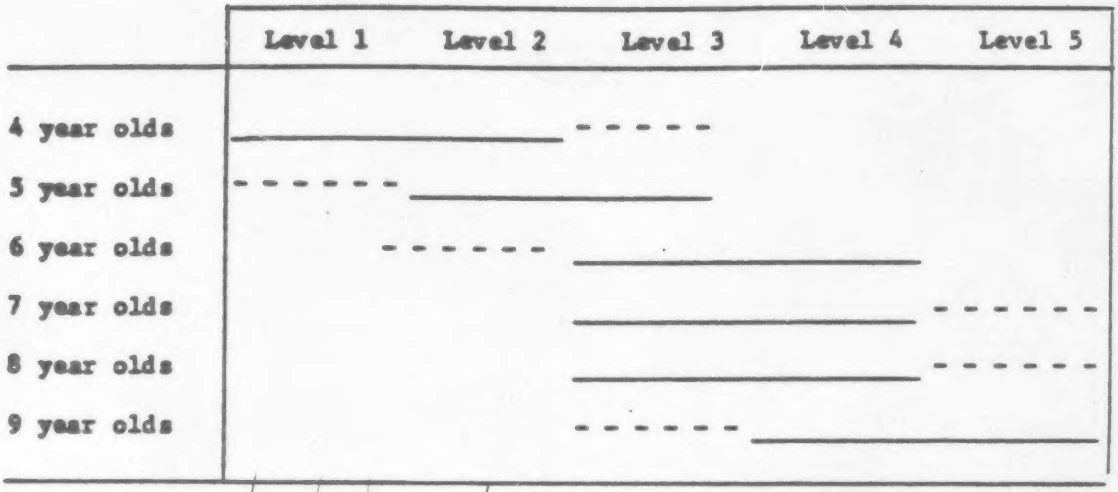


Figure 11. An ordinal description of children's ideas concerning the meaning of digits and numerals.

Portions of this study dealing specifically with ideas that children bring to bear on digits and numerals was replicated in another community (Cambridge). The children were drawn from a lower income population than were the subjects who formed the population sampled for this study. The children ranged in age from five to thirteen, and came from all grade levels from kindergarten through the sixth grade. The range of ideas expressed by the Cambridge children were no different than those reported in this study. The major difference between the samples in the two studies rested in the level x age analysis: the middle class suburban children's levels in each age bracket were slightly higher than those found in the lower class urban sample of children. The fifth and sixth graders in Cambridge responded in much the same fashion as the fourth graders in Belmont. This finding is consistent with many other developmental studies.

The utility of this study for mathematics education lies primarily in the documentation of children's ideas concerning the relationship between notational marks and numerical quantities, in contexts other than the ones they have encountered in the classroom. Children may demonstrate a very adequate understanding of place value if that understanding is assessed by means of the same kinds of lessons that they have been taught in school. But for many children this understanding is in fact limited to the particular kinds of examples or cases that they have already learned.

The place value property of the notational system is difficult for children to grasp for a number of reasons. Among them the following seem to be particularly important. First, the positional feature is one aspect of our written system, and it has to be understood in relation to other properties such as the use of zero to hold the place of an empty position, or to mark the absence of ones, groups of ten, and so forth. Second, the numerical ideas underlying place value seem to require a second order understanding of multiplicative relations, namely the representation of the number of times a group (or groups) of ten is written. Third, the purpose for which place value is taught, that is regrouping for addition and subtraction, is an algorithm or procedure that can be carried out without reference to the numerical ideas underlying place value. Column addition and subtraction can be performed by treating each and every place as if it were the same, as if the numerical referents were unimportant. Many children as well as adults carry out this procedure and arrive at accurate results without a moment's concern about the numerical ideas represented by position, that the position in which a digit occurs determines its value.

Appendix A:

Samples of Children's Drawings

Symbolic Representation: Six Sticks

Level 1 183

Level 2 187

Level 3 189

Symbolic Representation: Wheels and Cars

Level 1 192

Level 2 194

Level 3 197

Level 4 199

Symbolic Representation: Packs of Gum

Level 1 207

Level 2 208

Level 3 209

Level 4 210

Conventional Representation: Writing Numerals[*]

Level 1 217

Level 2 218

Level 3 223

Conventional Representation: Meanings of Digits and Numerals

Level 1[**] 227

Level 2 229

Level 3 232

Level 4 244

Level 5[***] 251

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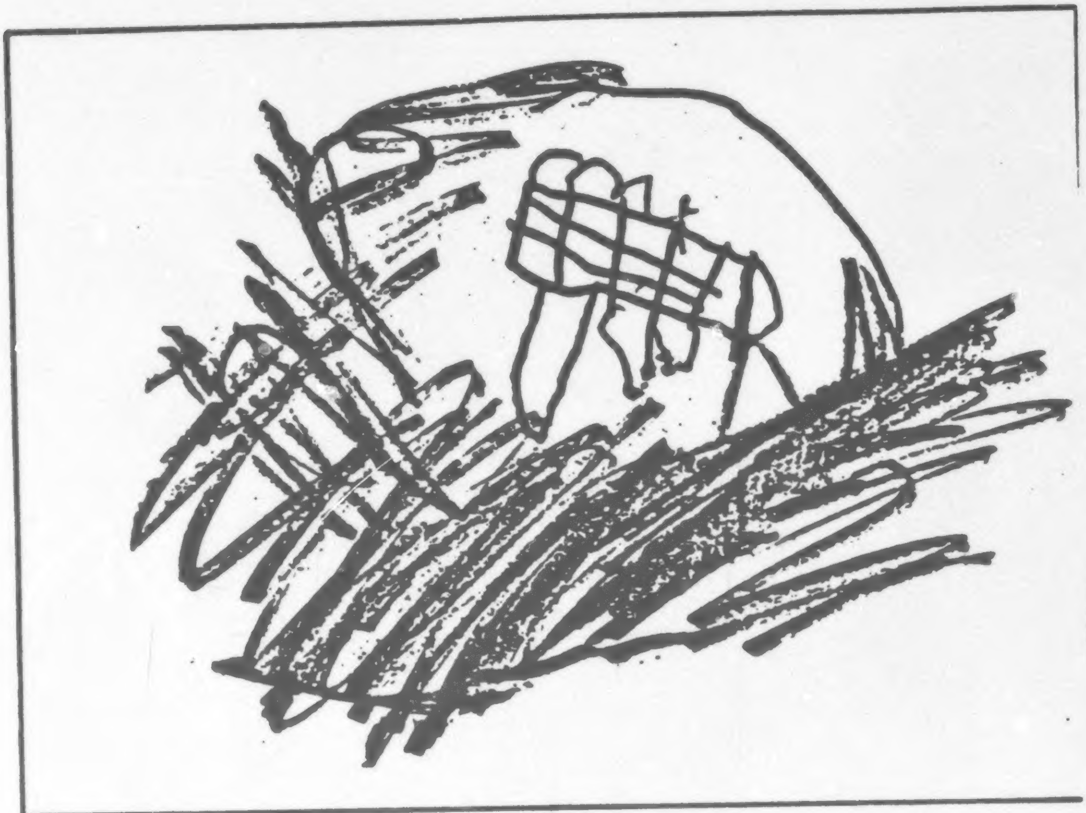
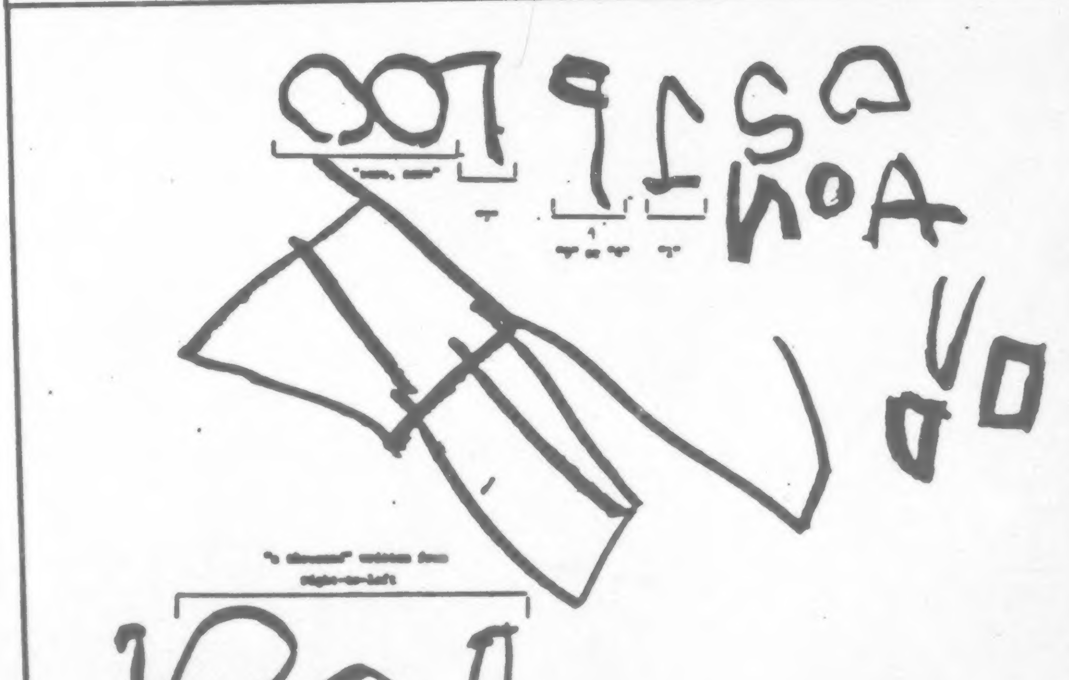
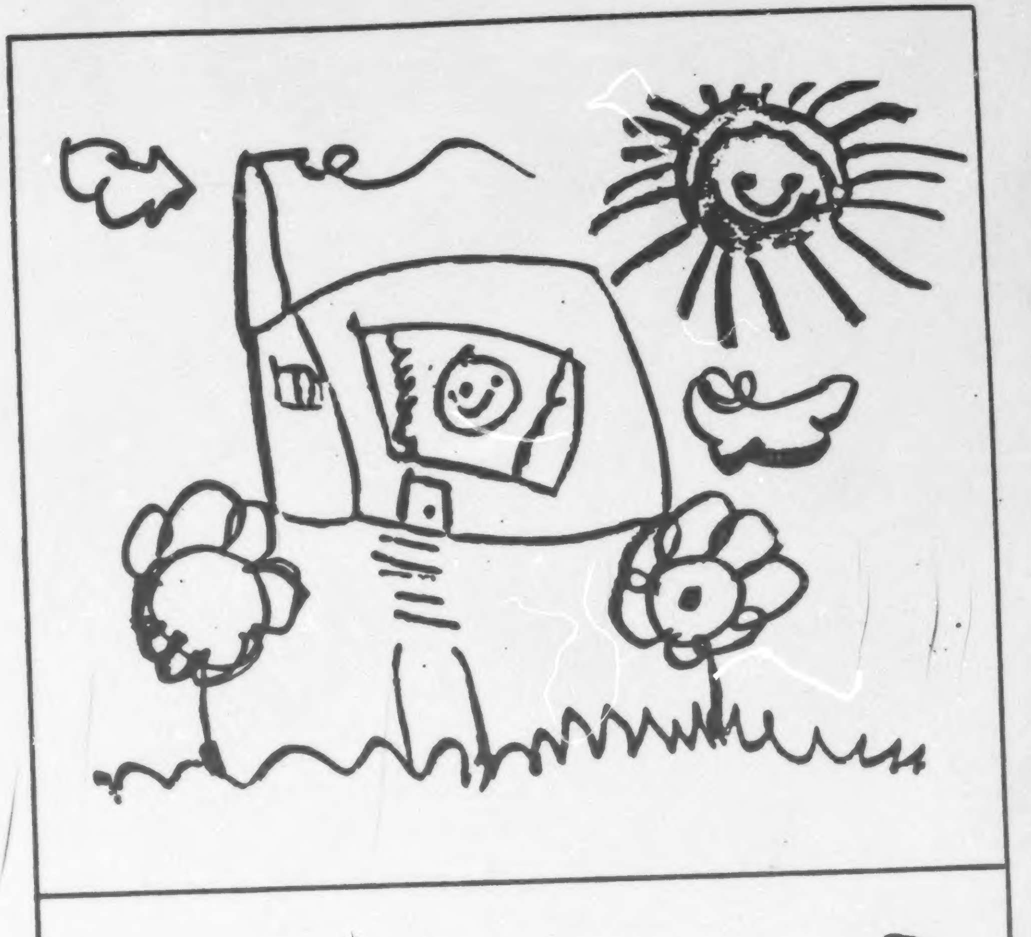
[*] Level 4 is not necessary to include.

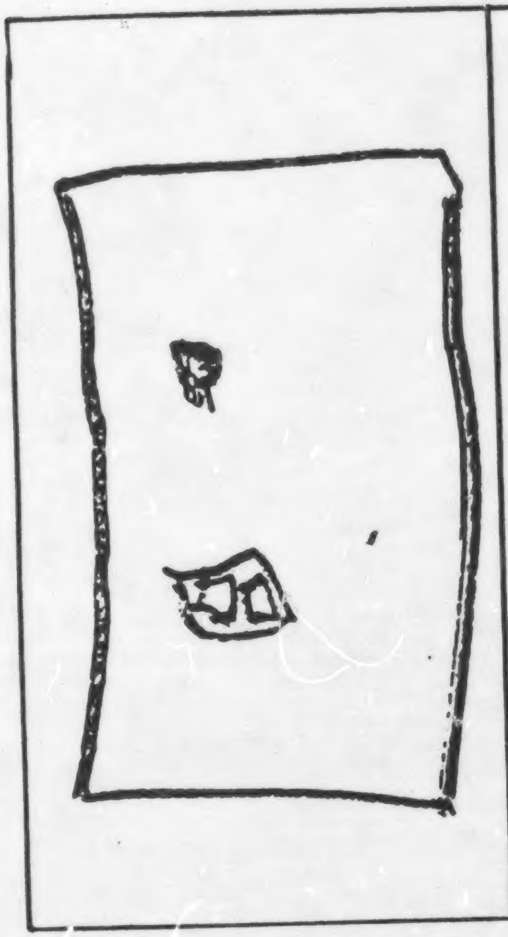
[**] See also p. 219.

[***] See also pp. 203-206 for Level 5 on Wheels task, and pp. 212-216 for Level 5 on Gum task.

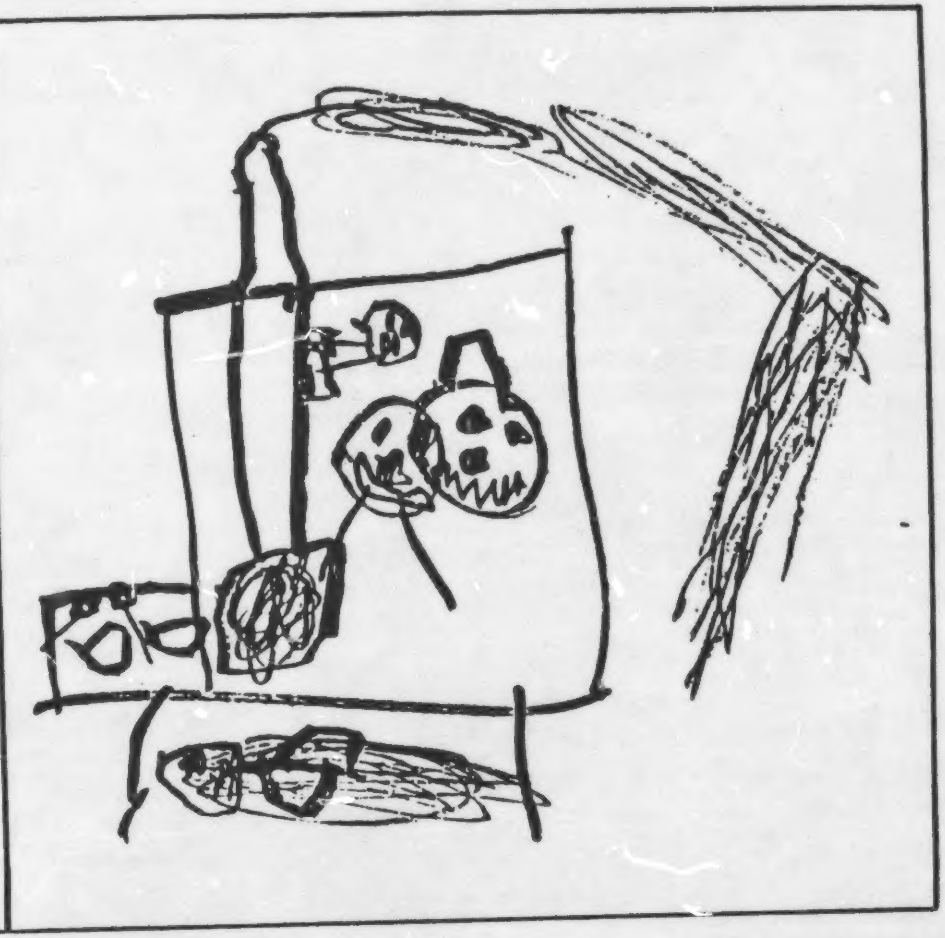



"a thousand" written from
right-to-left

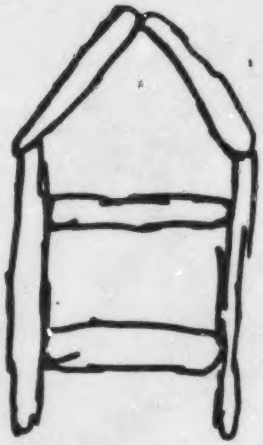


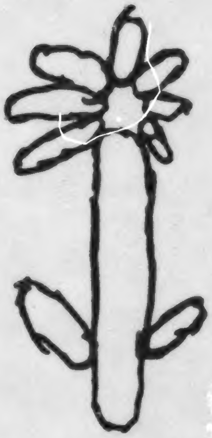


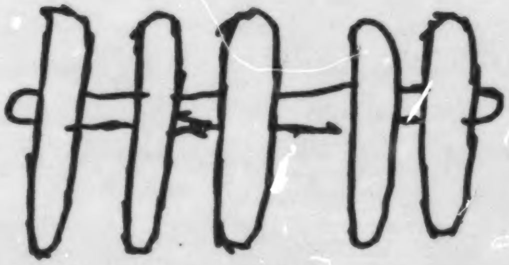
Drawing in response to request for "some way to remember" 4-and-2 arrangement of sticks.

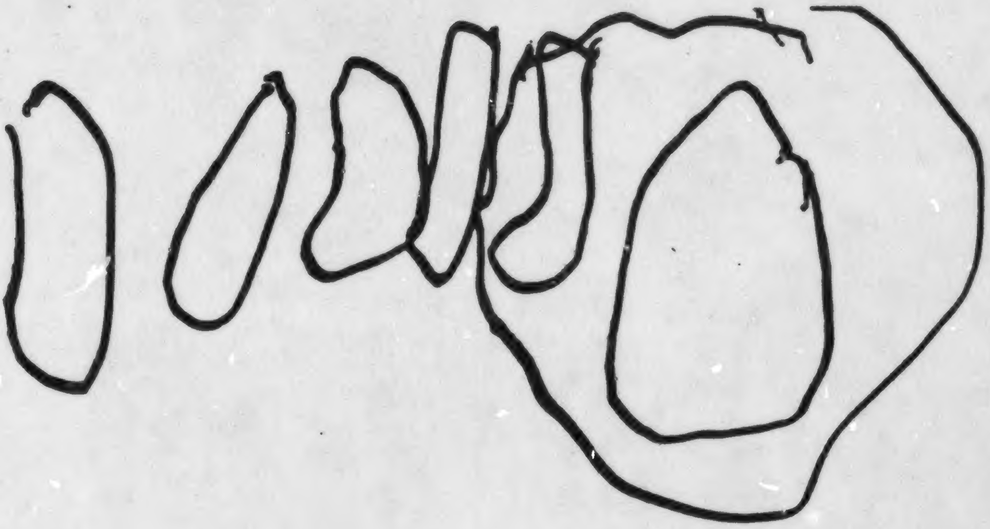


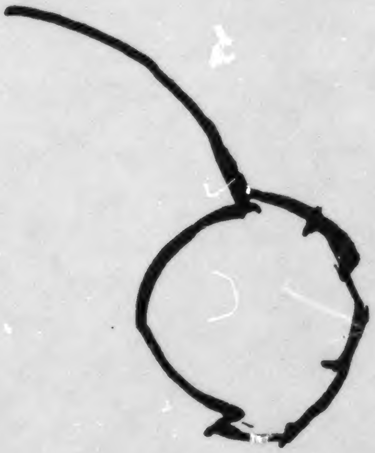
Drawing in response to request for "some way to remember" 3-and-3 arrangement of sticks.

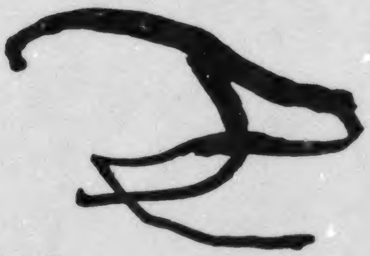


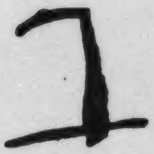


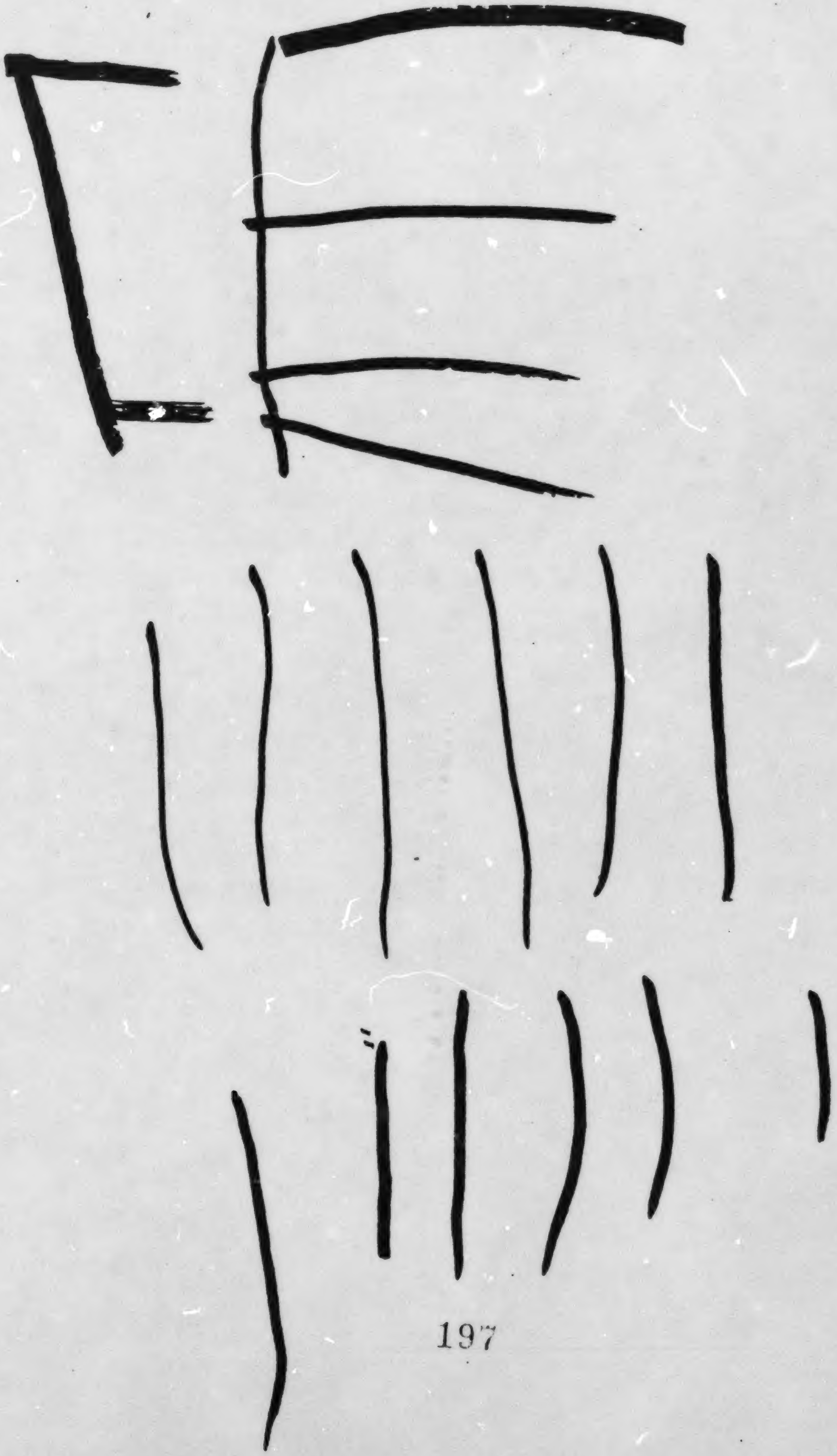


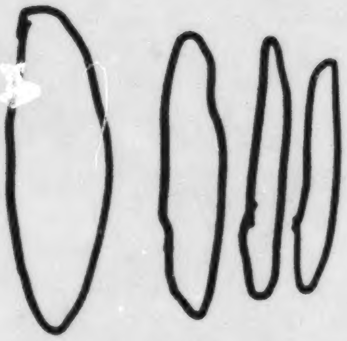


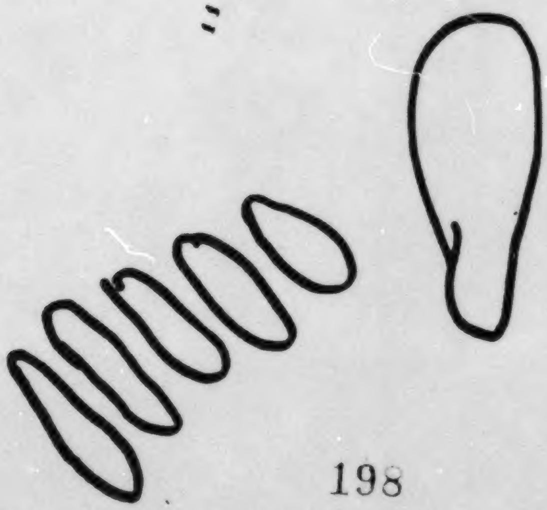


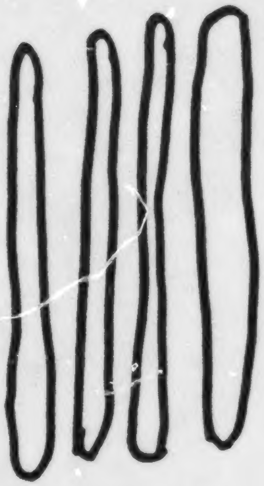


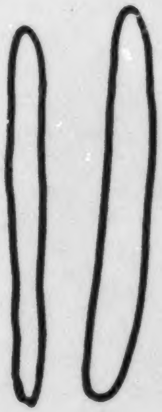


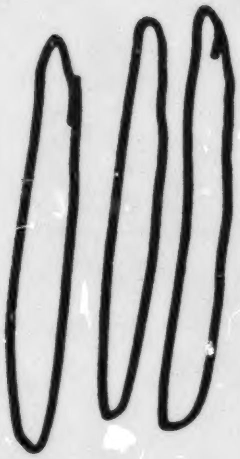


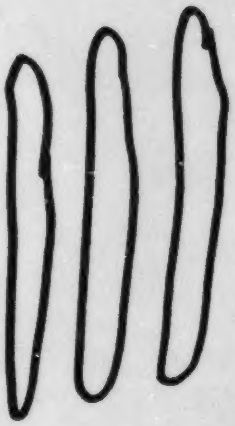


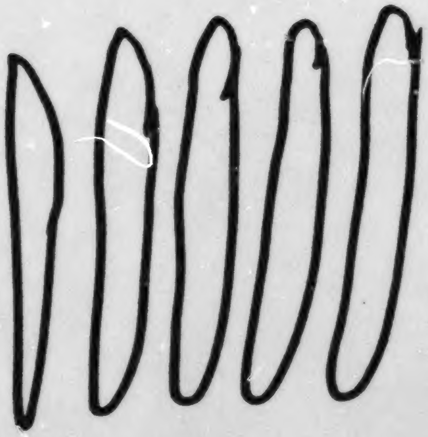


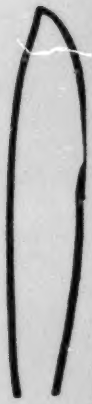












2 on one side.

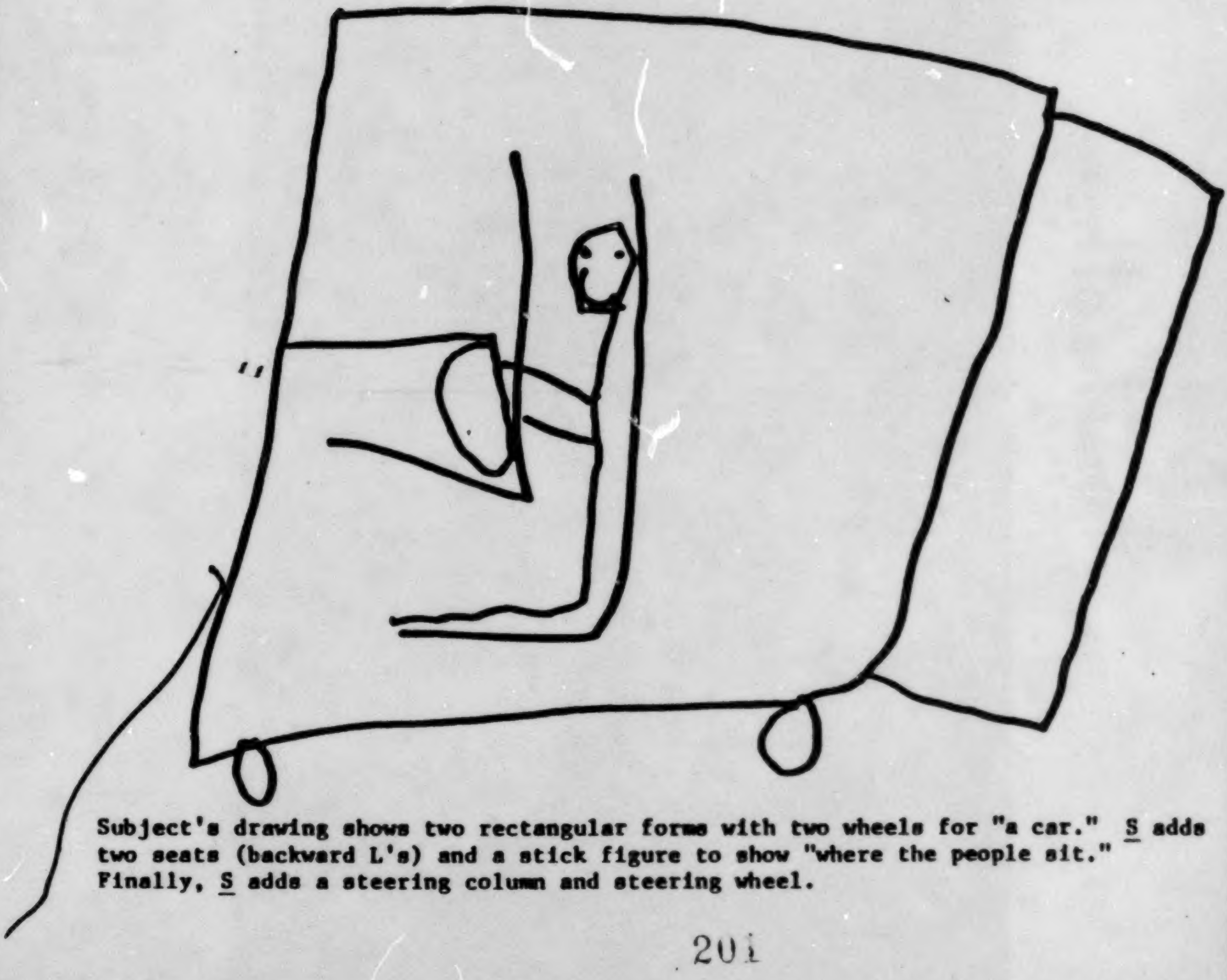
4 on the other.

3 on one side.

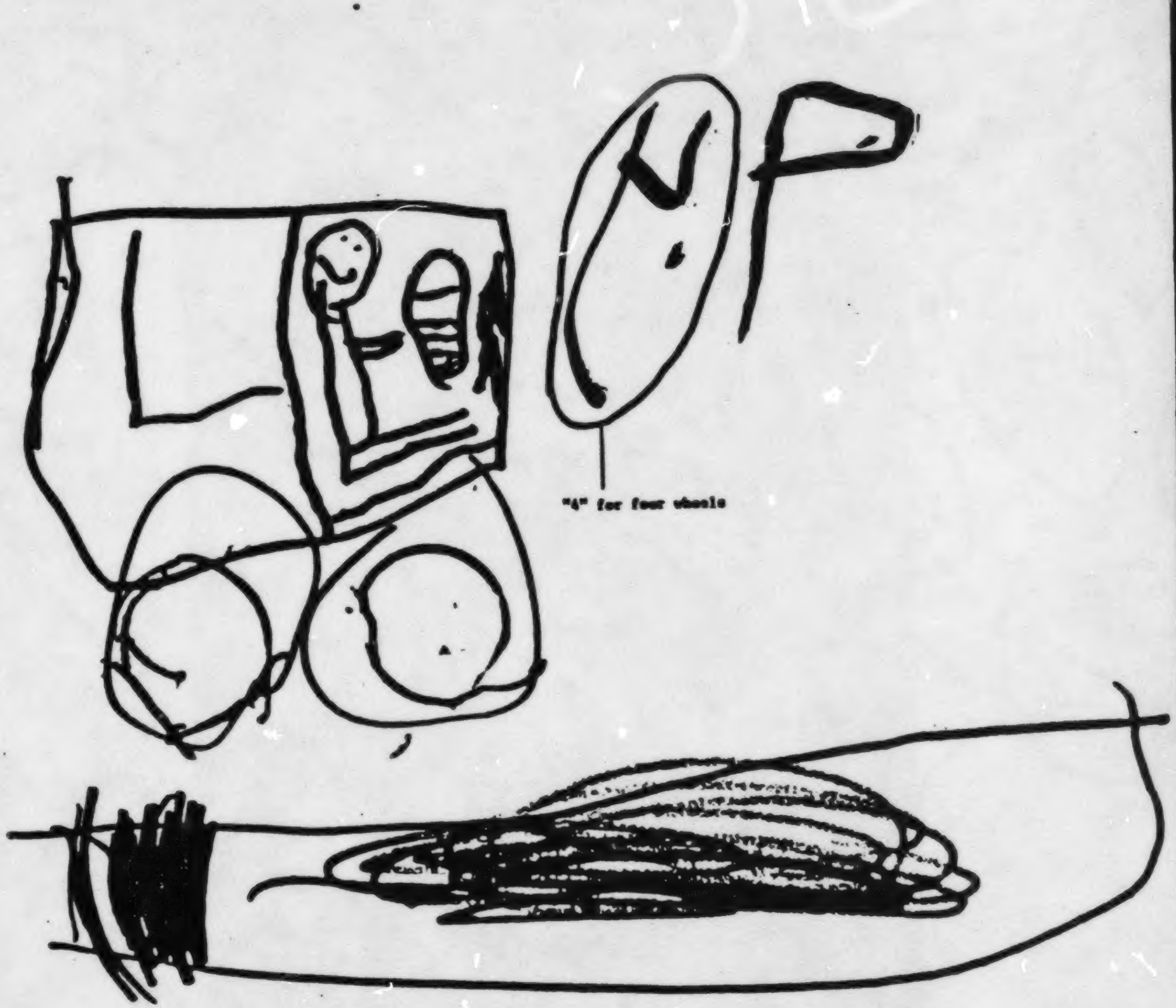
3 on the other.

1 on one side.

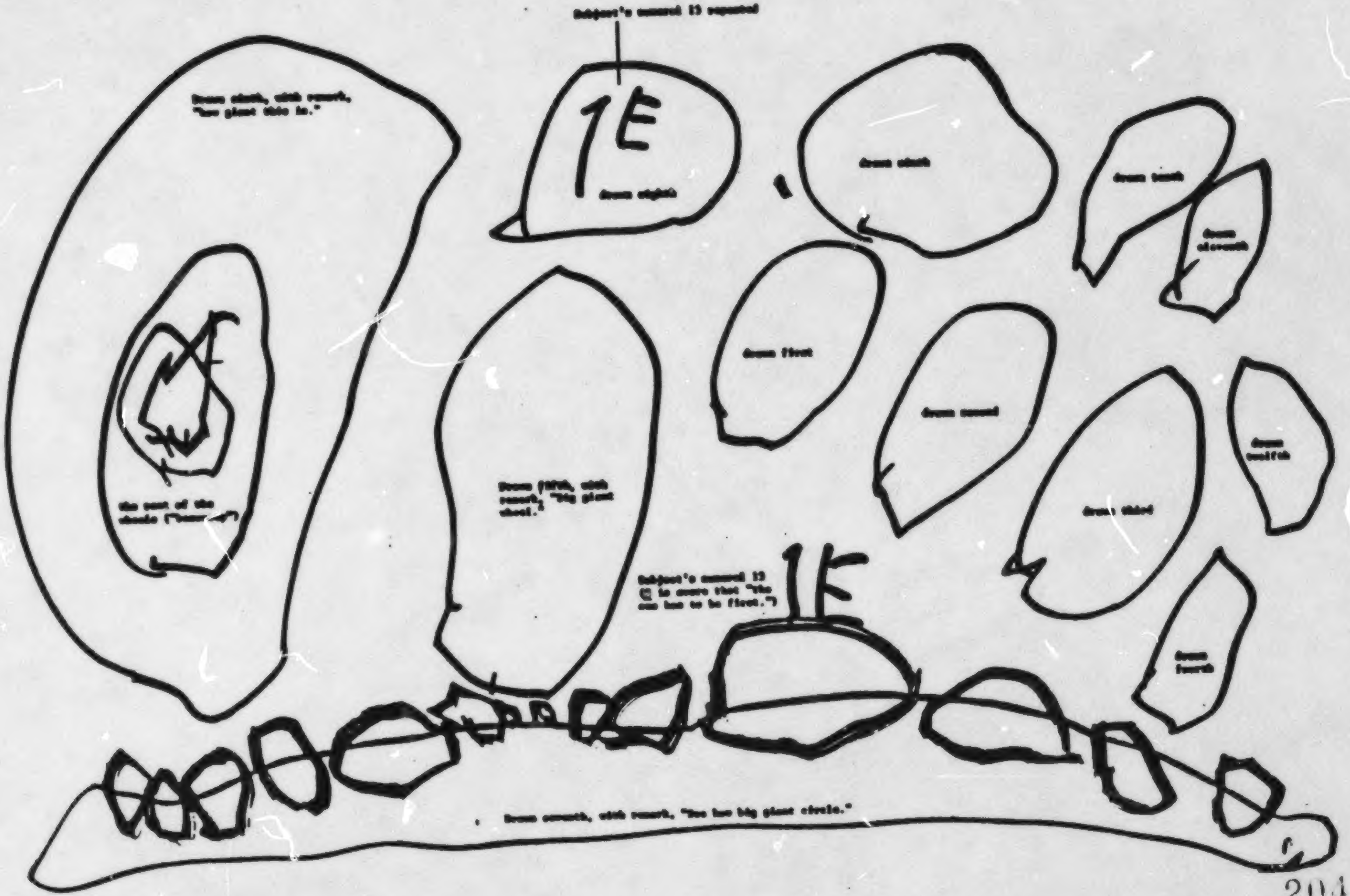
5 on the other.



Subject's drawing shows two rectangular forms with two wheels for "a car." S adds two seats (backward L's) and a stick figure to show "where the people sit." Finally, S adds a steering column and steering wheel.



"4" for four wheels



Subject's numeral 13 repeated

Drawn ninth, with remark, "Too giant this is."

Drawn eighth

Drawn sixth

Drawn tenth

Drawn eleventh

the rest of the circle ("[illegible]")

Drawn fifth, with remark, "big giant wheel."

Drawn first

Drawn second

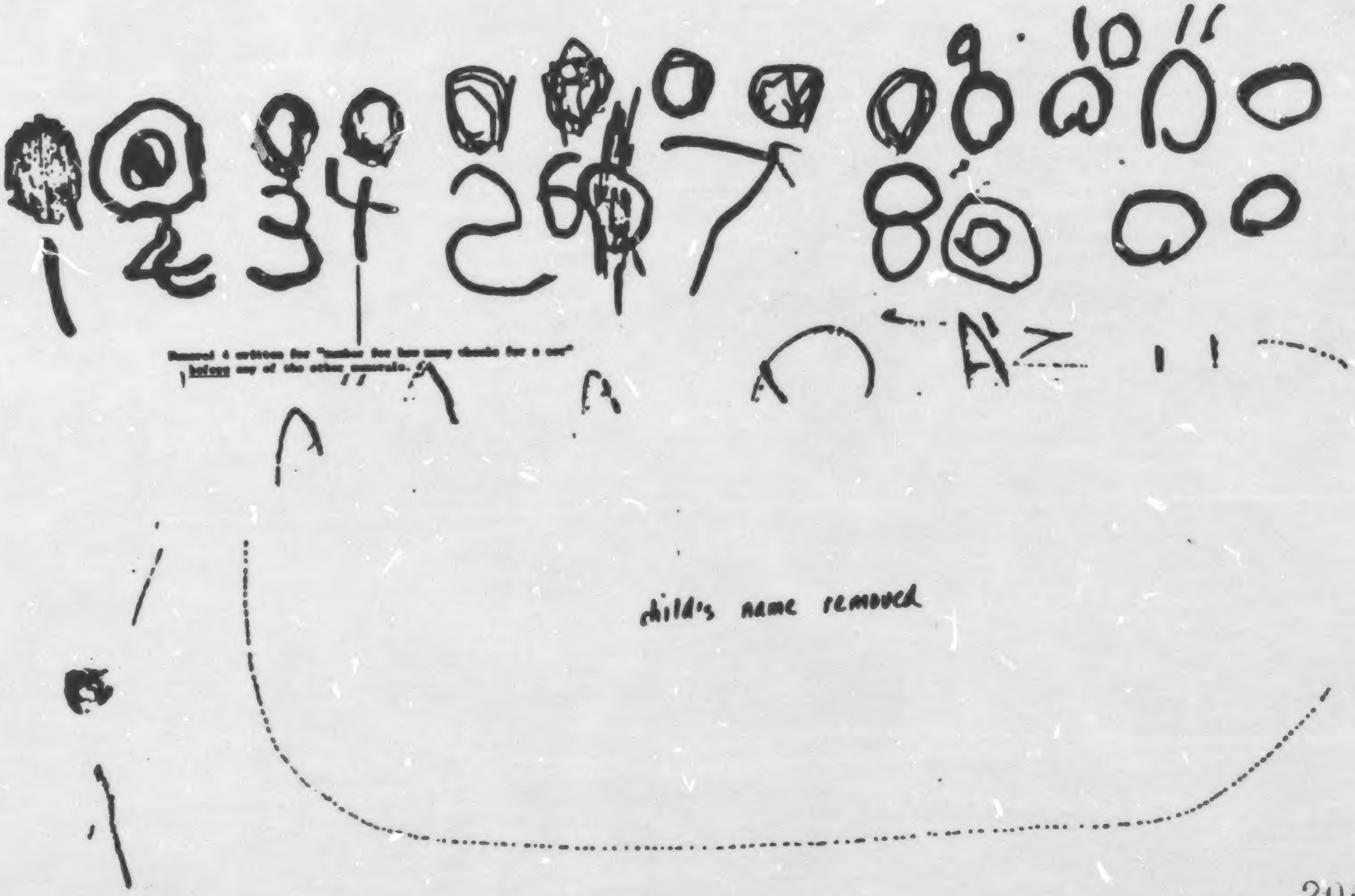
Drawn twelfth

Drawn third

Subject's numeral 13 (E) to assure that "the one has to be first.")

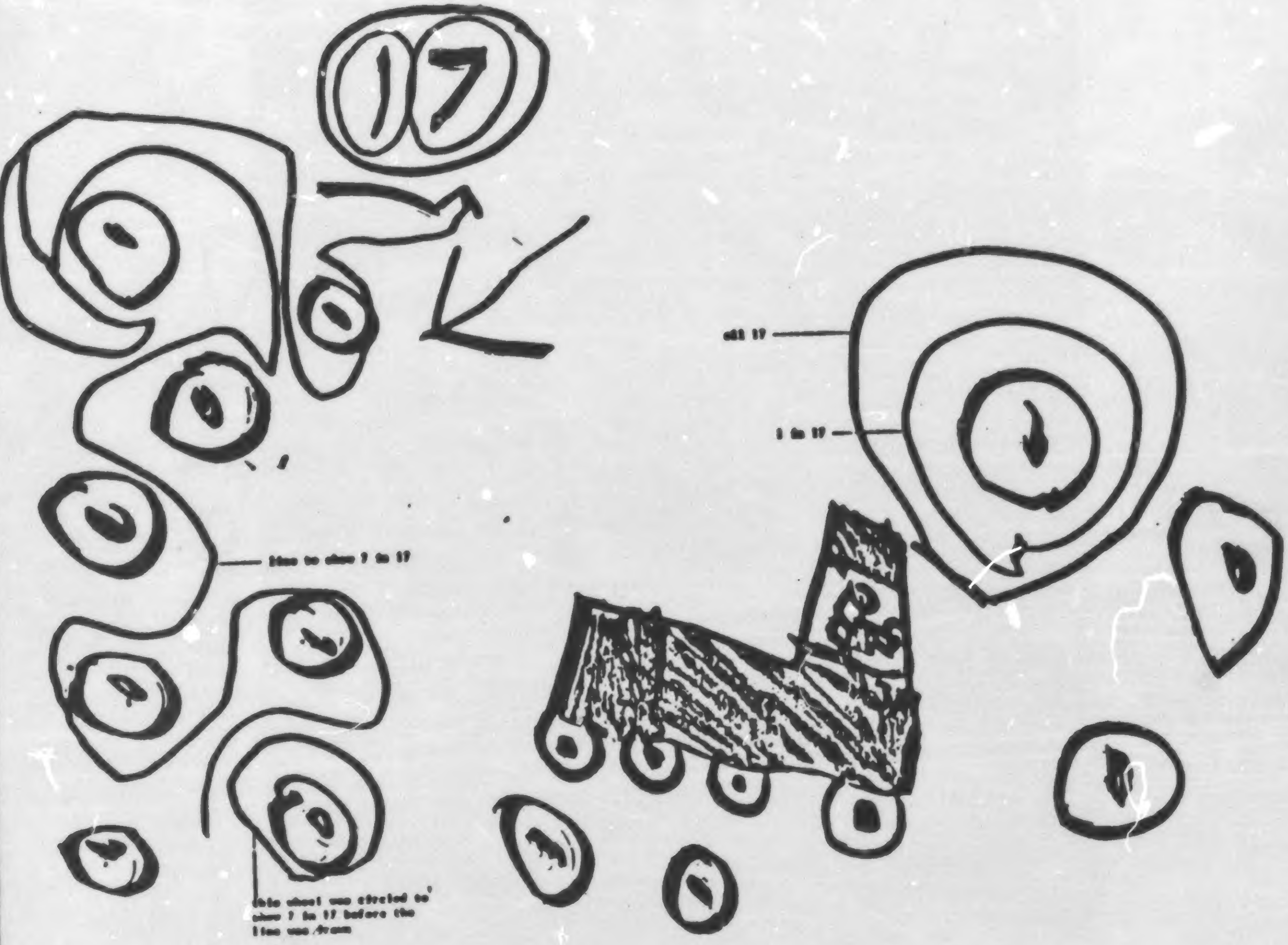
Drawn fourth

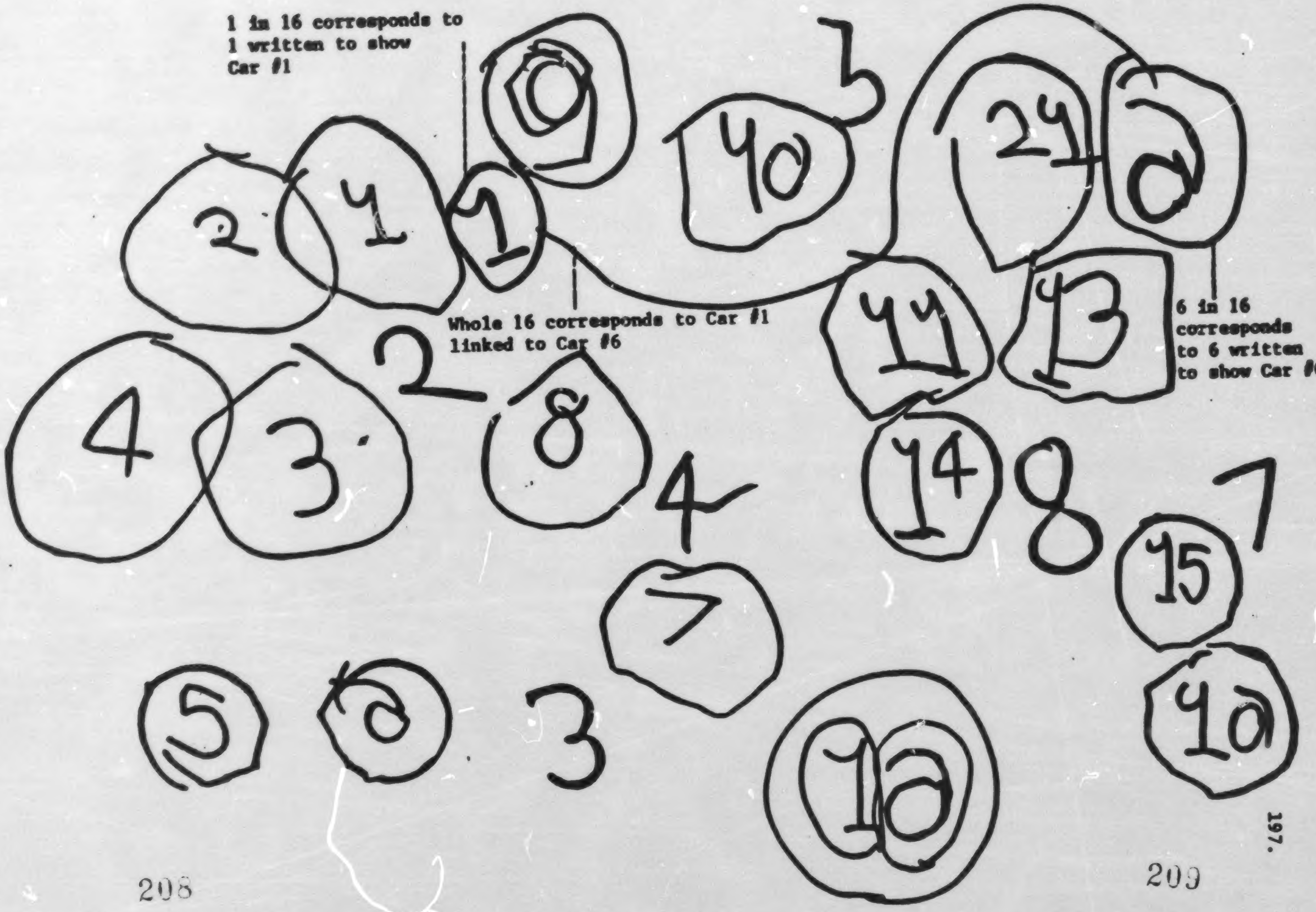
Drawn seventh, with remark, "Too too big giant circle."


1 2 3 4 5 6 7 8 9 10 11

Numeral 4 written for "number for how many wheels for a car" before any of the other numerals.

child's name removed

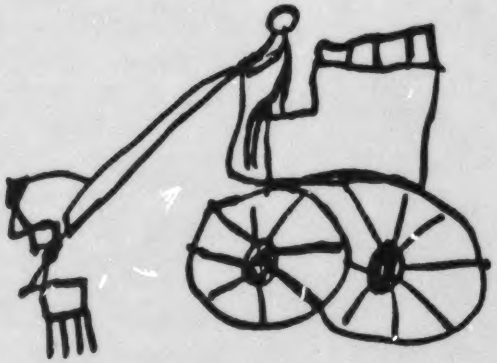

17
all 17
1 in 17
line to show 7 in 17
this wheel was circled to show 7 in 17 before the line was drawn

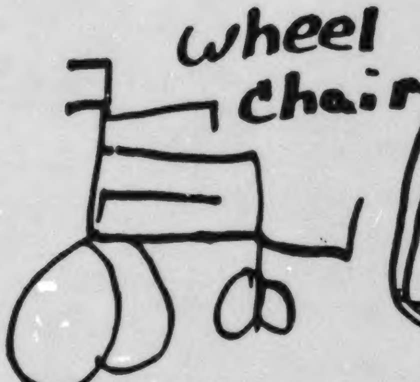

2
4
1
9
40
3
24
2
44
13
4
3
2
8
4
14
8
7
15
7
5
2
3
12
12

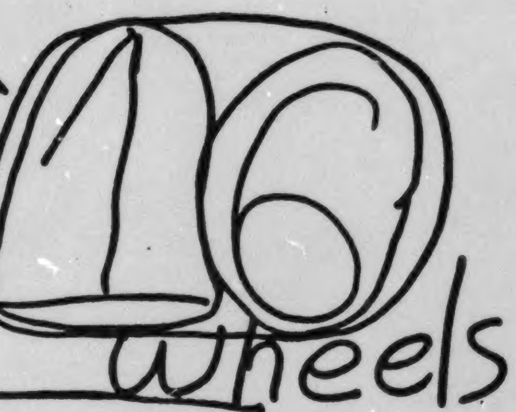
1 in 16 corresponds to 1 written to show Car #1

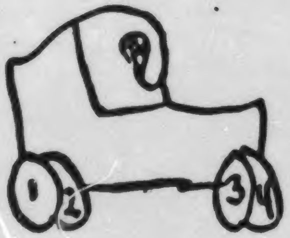
Whole 16 corresponds to Car #1 linked to Car #6

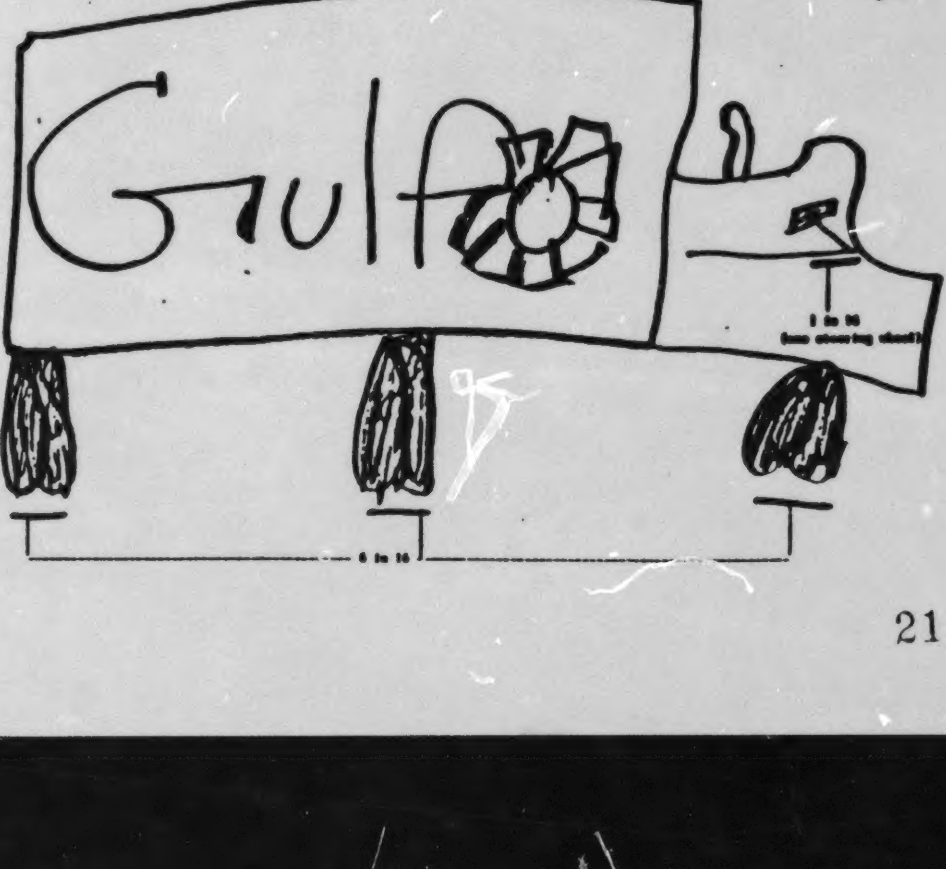
6 in 16 corresponds to 6 written to show Car #6



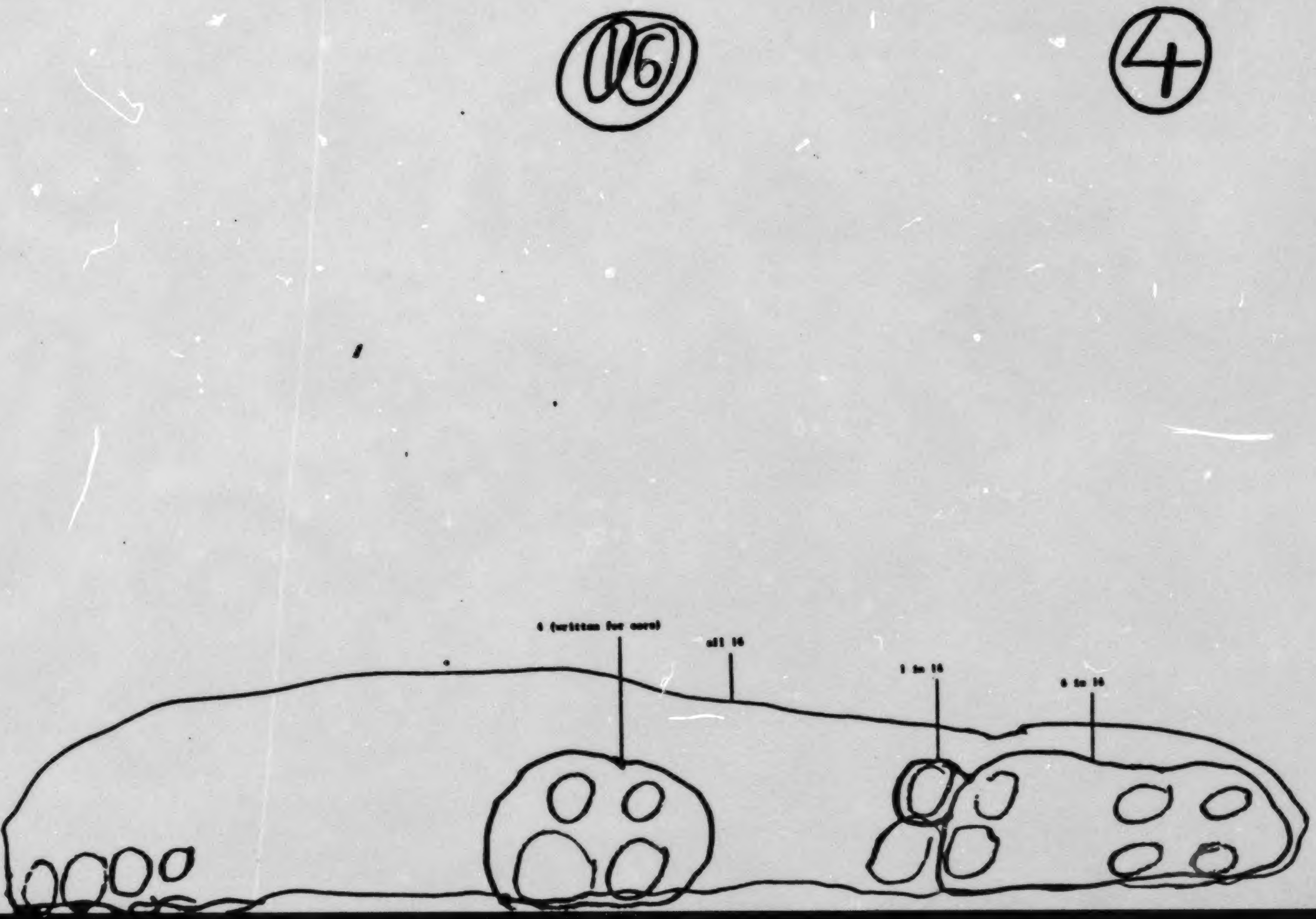

wheel chair


10 wheels

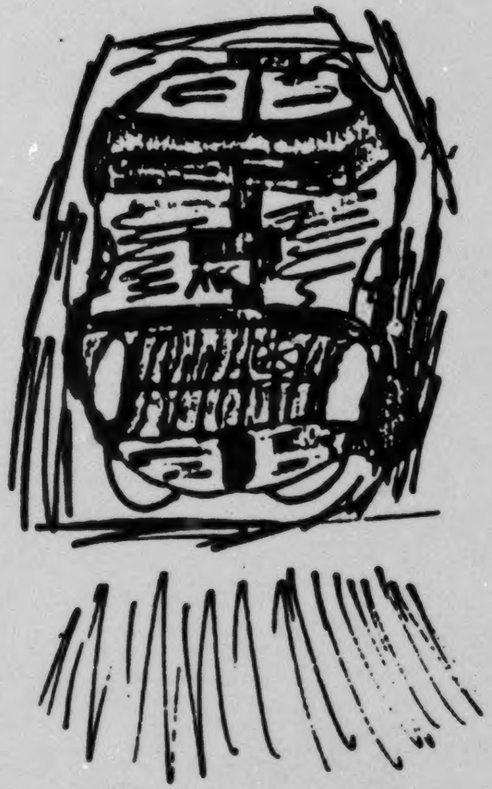

1
2
3
4


Gulf

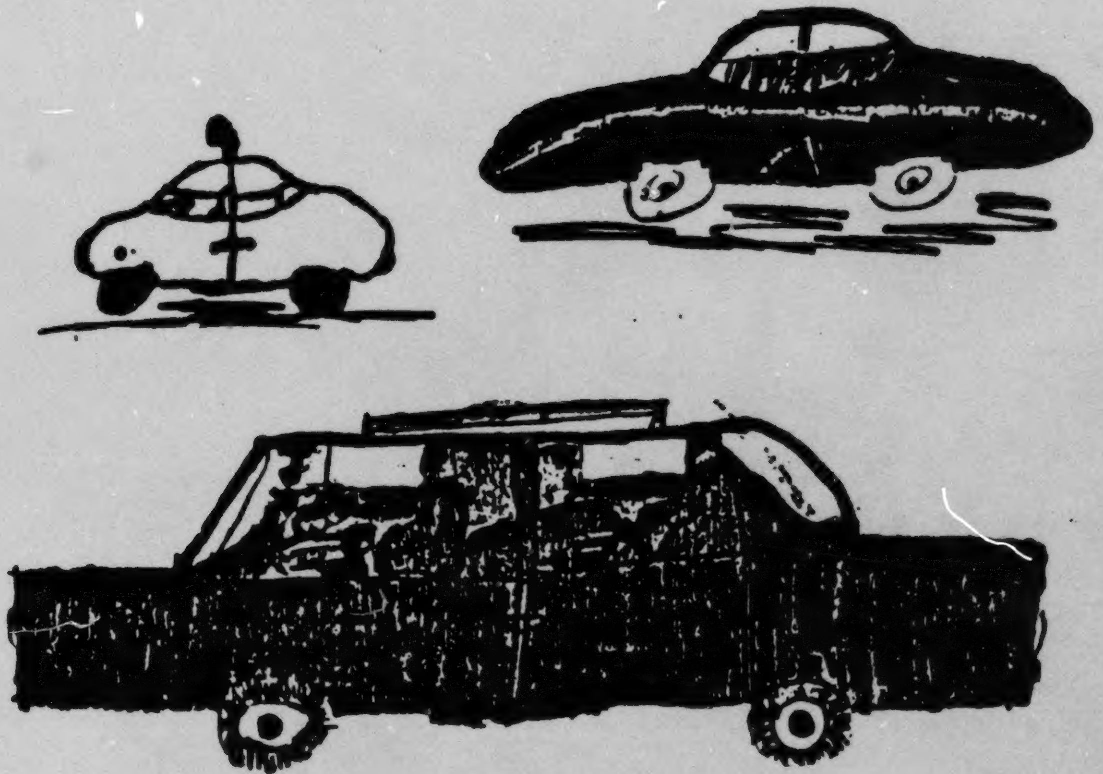
212


16
4
4 (written for zero)
all 16
1 in 16
4 in 16

213

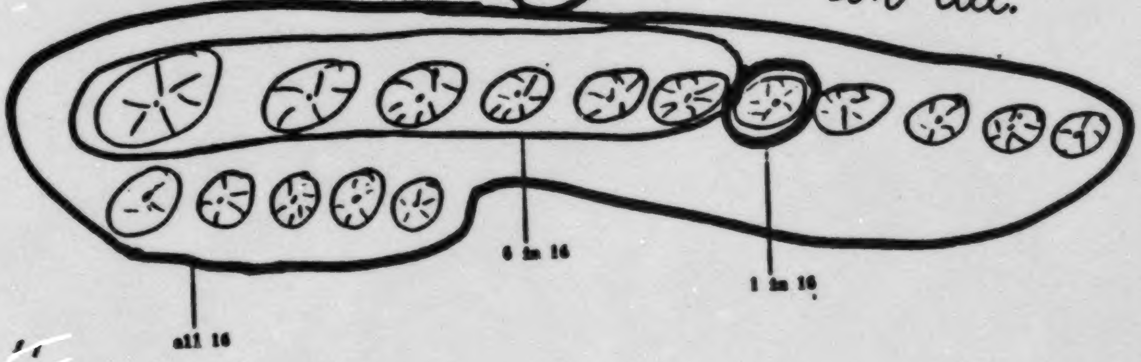


214



215

there are 4 cars in my pichture.
And there are 16 wheels in all.

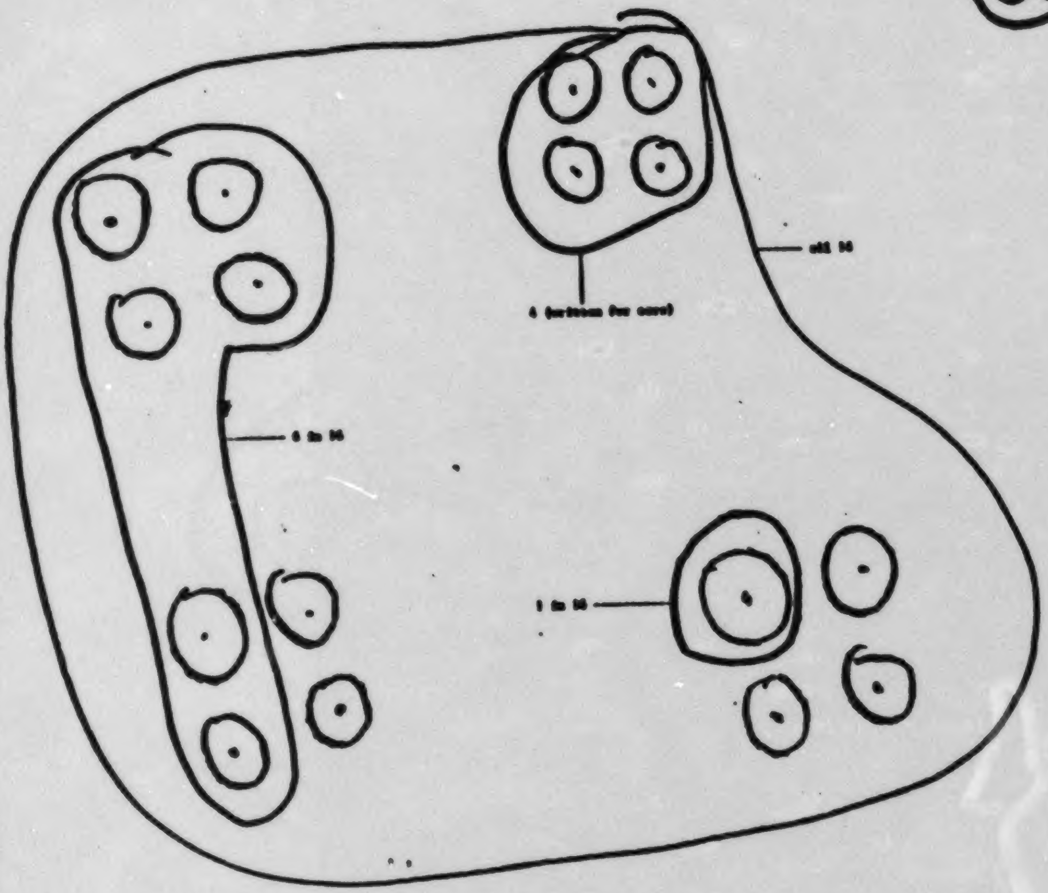


the six means. ones in the 16 wheels.

the green line means that there are 16 wheels.

the 1 means tens in the 16 wheels.

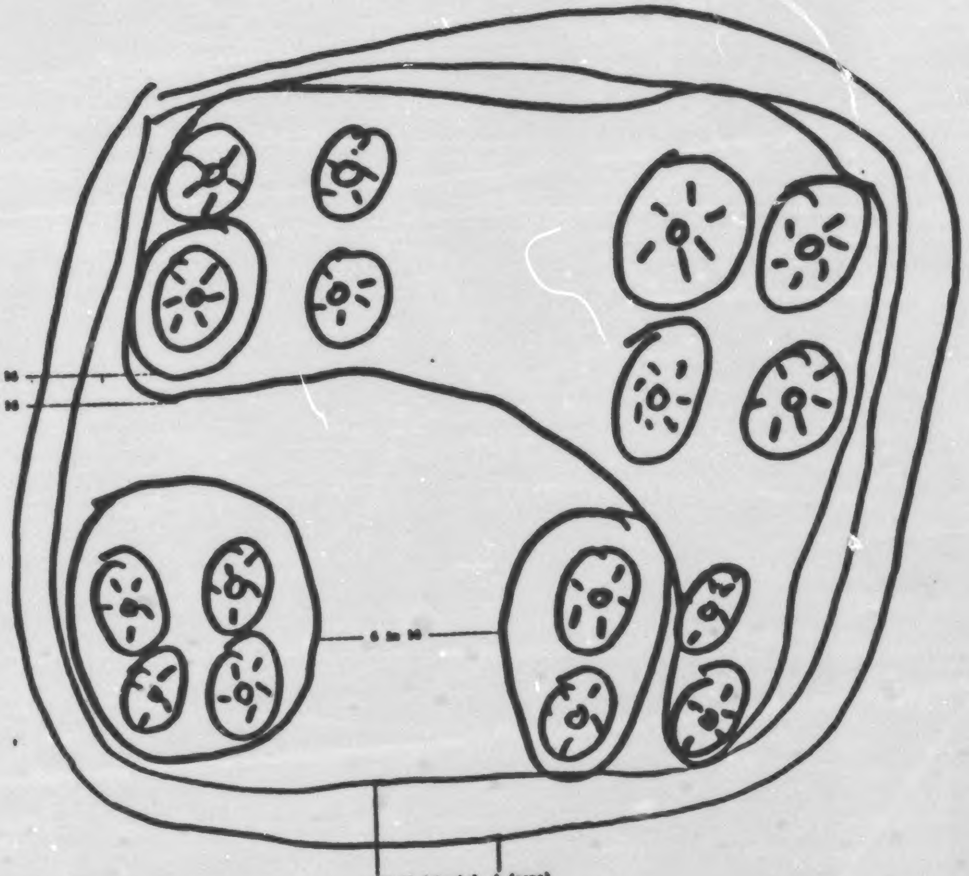
16

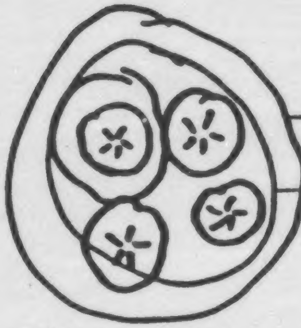

all 16
4 (burdens for one)
6 in 16
1 in 16

4

$$\begin{array}{r} 8 \\ +\,8 \\ \hline 16 \end{array}$$

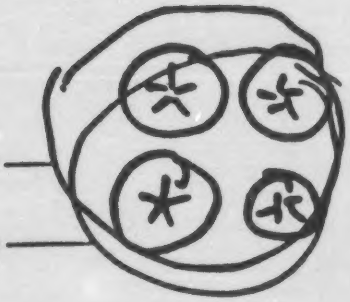
4 groups of cars

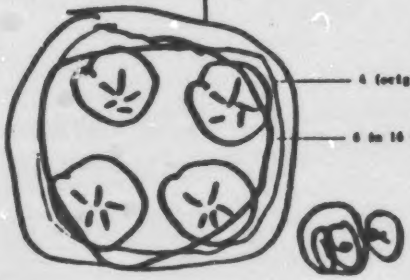

first circled one wheel for 1 in 16
then circled two wheels for 1 in 16
4 in 16
all 16 (wheels)
4 (cars)

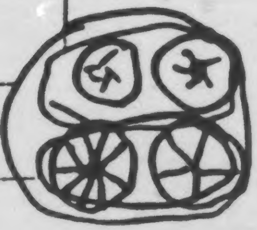

one of four circles drawn for all 16
1 in 16 = 8 + 6 + 2

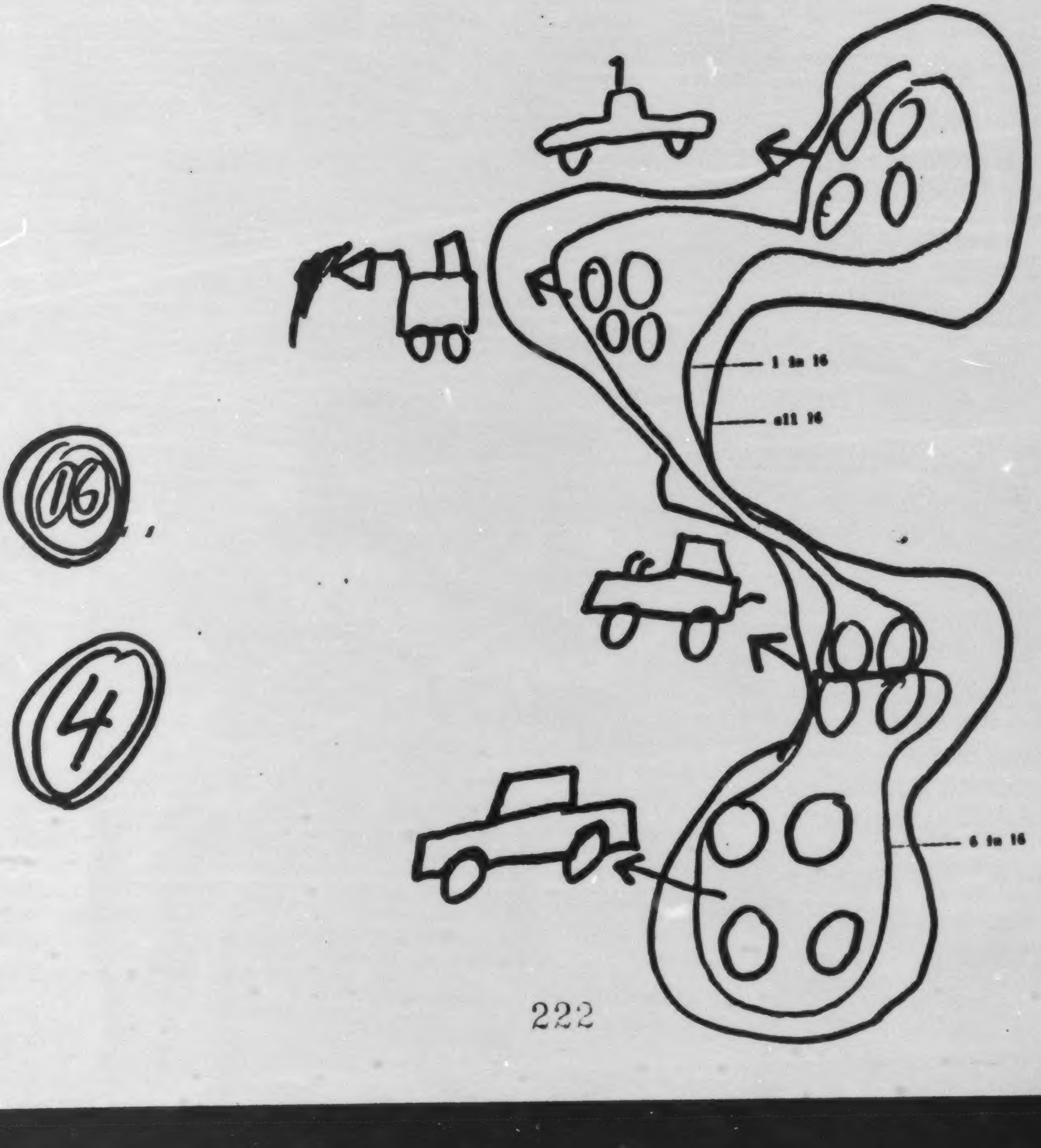
Subject initially denoted 1 in 16 as "one wheel."

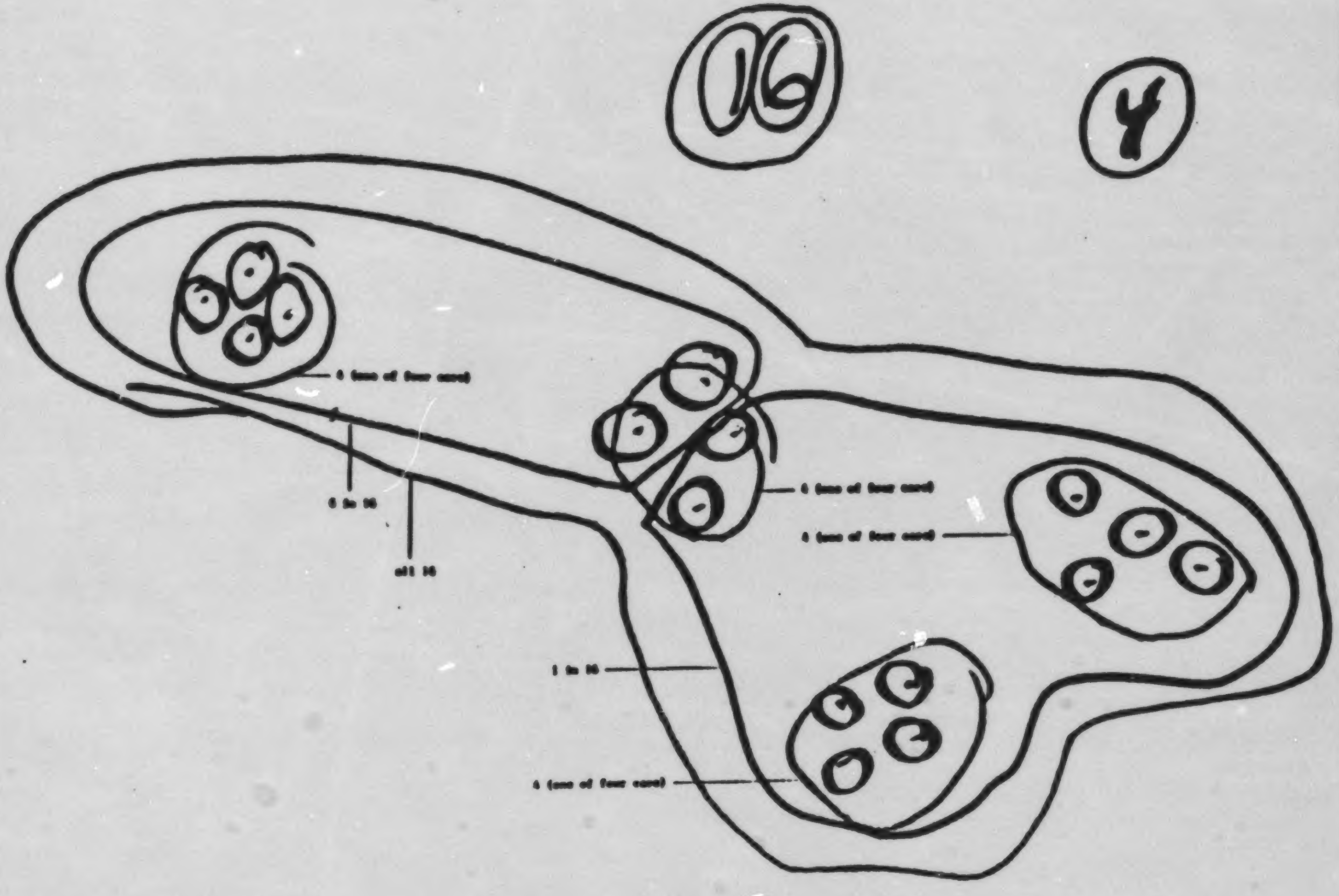
After doing Task 5, Version 3 (goal in which the 2 in 23 was immediately seen as meaning twenty, the S saw error and circled two wheels for the 1 in 16.

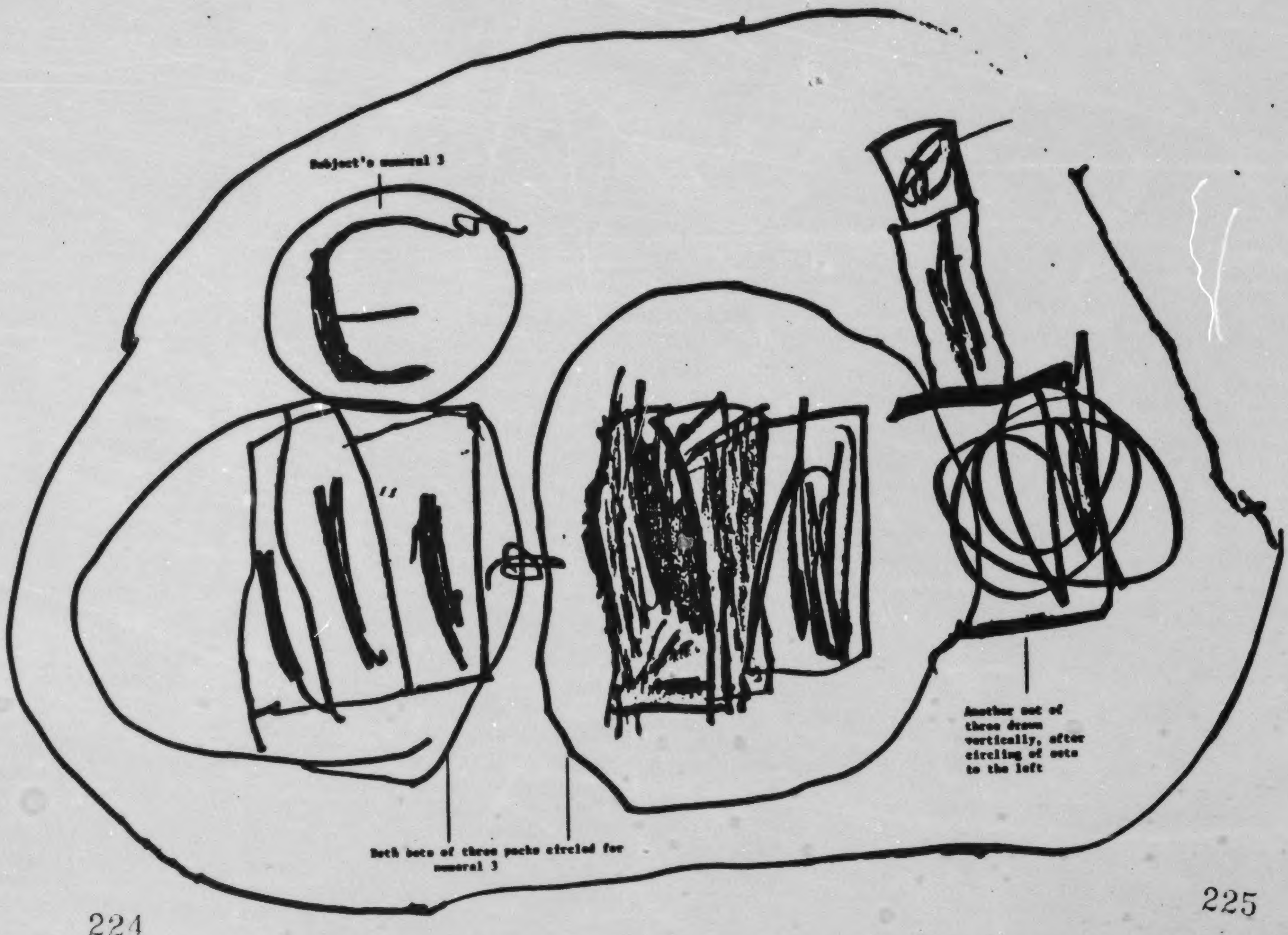

one of four circles drawn for all 16
1 in 16 = 8 + 6 + 2


one of four circles drawn for all 16
4 (originally written for number of cars)
6 in 16 + 8 + 2


1 in 16 = 8 + 6 + 2
one of four circles drawn for all 16
6 in 16 = 8 + 2


1
1 in 16
all 16
6 in 16
16
4


16
4
4 (one of four core)
6 in 16
all 16
4 (one of four core)
4 (one of four core)
1 in 16
4 (one of four core)



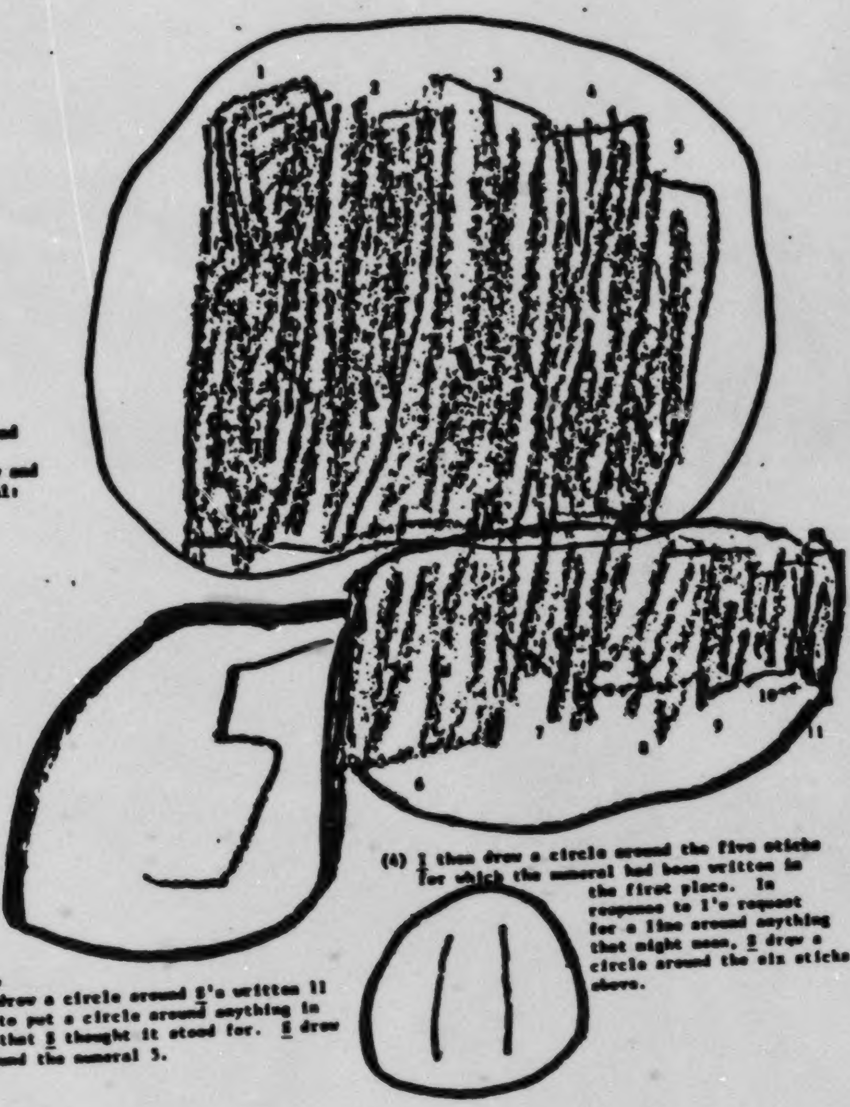
Subject's numeral 3

Another set of three drawn vertically, after circling of sets to the left

Both sets of three packs circled for numeral 3

224

225

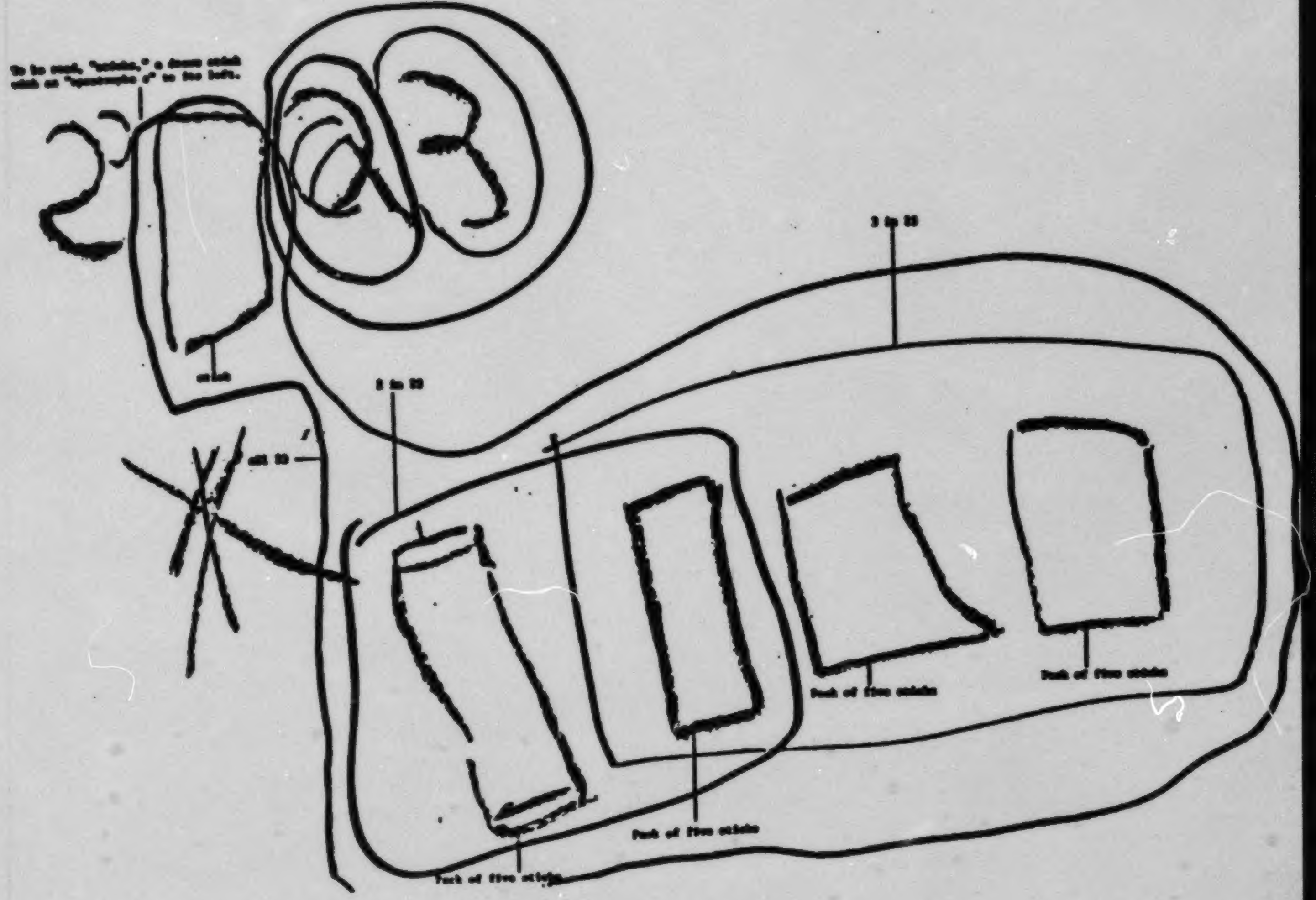


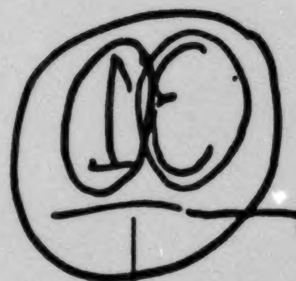
(1) Subject drew five sticks of gum and then wrote the numeral 5.

(2) S then drew six more sticks below and wrote the numeral 11 for the total: "Eleven, eleven I drawed."

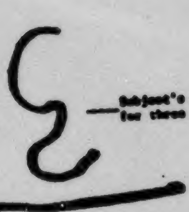
(3) Interviewer drew a circle around S's written 11 and asked S to put a circle around anything in the picture that S thought it stood for. S drew a circle around the numeral 5.

(4) I then drew a circle around the five sticks for which the numeral had been written in the first place. In response to I's request for a line around anything that might mean, S drew a circle around the six sticks above.

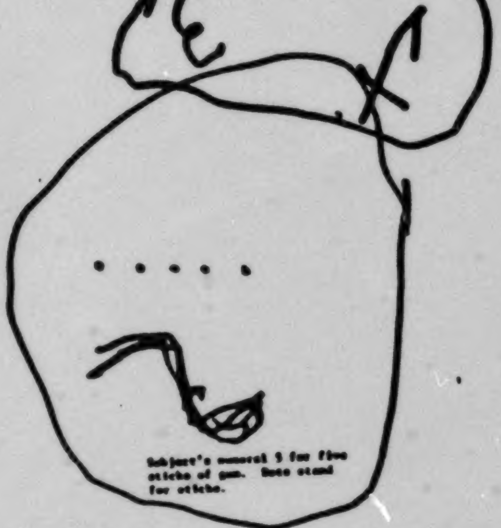

To be read, "dozens," a dozen stick with an "apostrophe s" to the left.
stick
Pack of five sticks
Pack of five sticks
Pack of five sticks
Pack of five sticks



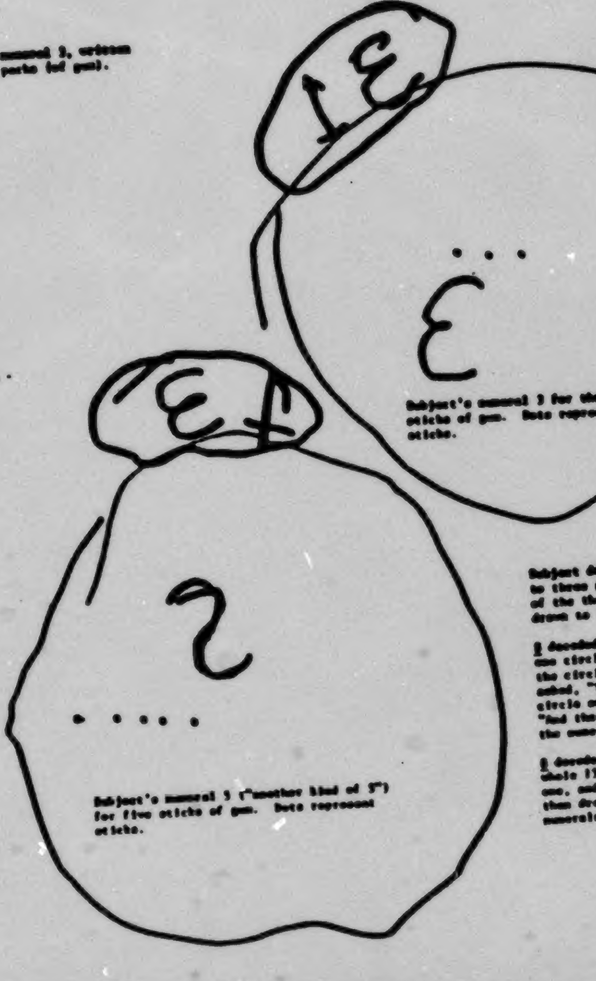
Subject's numeral 13: 3 was written first; the 1 followed. S asked about which side of the 3 the 1 should be written.



Subject's numeral 3, written for three packs (of gum).



Subject's numeral 5 for five sticks of gum. Dots stand for sticks.



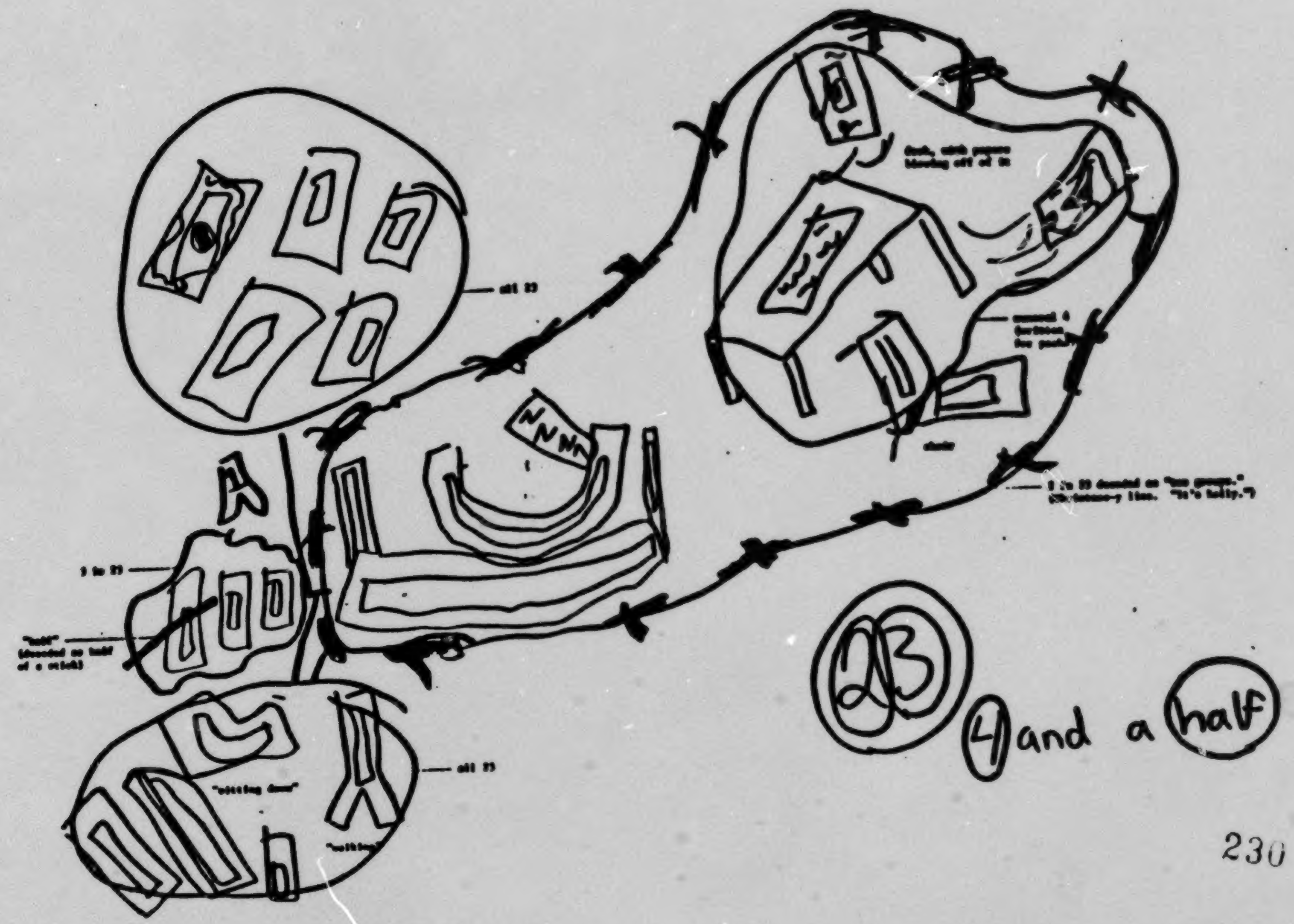
Subject's numeral 5 ("another kind of 5") for five sticks of gum. Dots represent sticks.

Subject's numeral 3 for three sticks of gum. Dots represent sticks.

Subject decoded 3 in 13 as referring to three packs. S wrote "3" on each of the three circles which had been drawn to represent packs of gum.

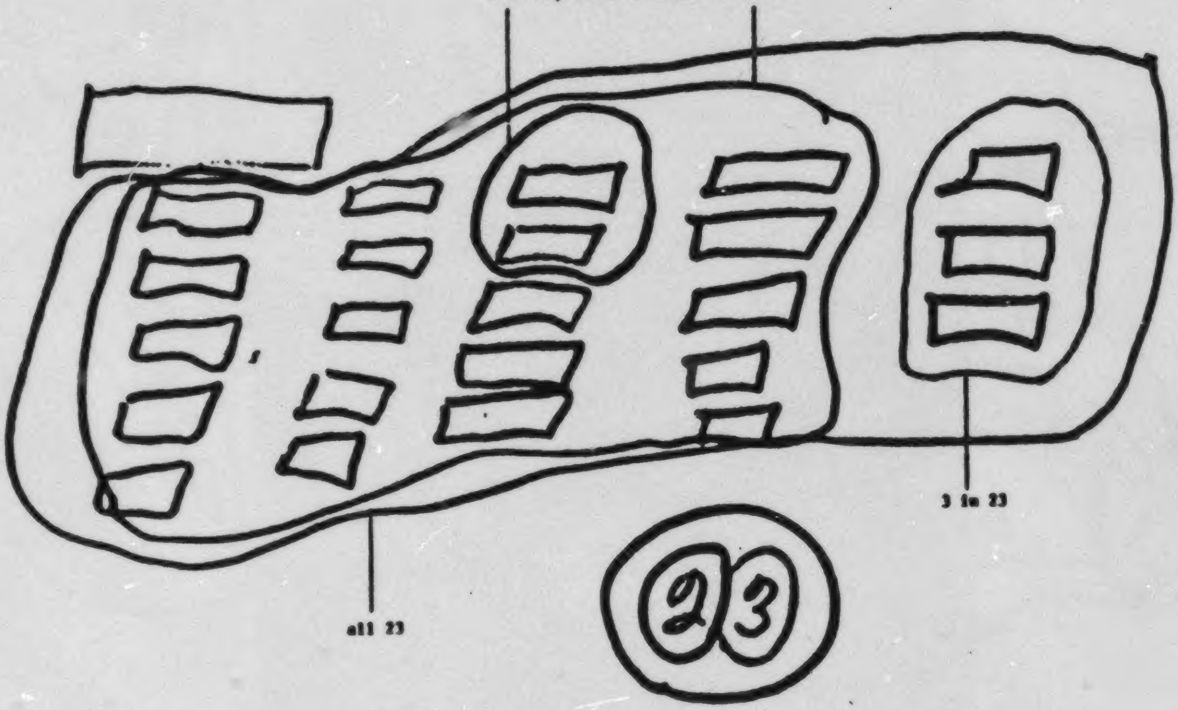
S decoded 1 in 13 as "There's each one circle." Interviewer pointed to the circles (drawn for packs) and asked, "There's one circle and one circle and one circle?" S responded, "And that makes three." S then wrote the numeral 1 on each of the circles.

S decoded the circle drawn around the whole 13 as: "There's one and one and one, and they all...make three." S then drew the three circles around the numerals lying on the pack-lines.


all 23
Desk, with papers blowing off of it
around 4 [illegible]
[illegible]
1 in 23 counted as "two people." ([illegible] line. "It's [illegible].")
3 in 23
"half" (counted as half of a child)
"sitting down"
"walking"
all 23
23
4 and a half

229

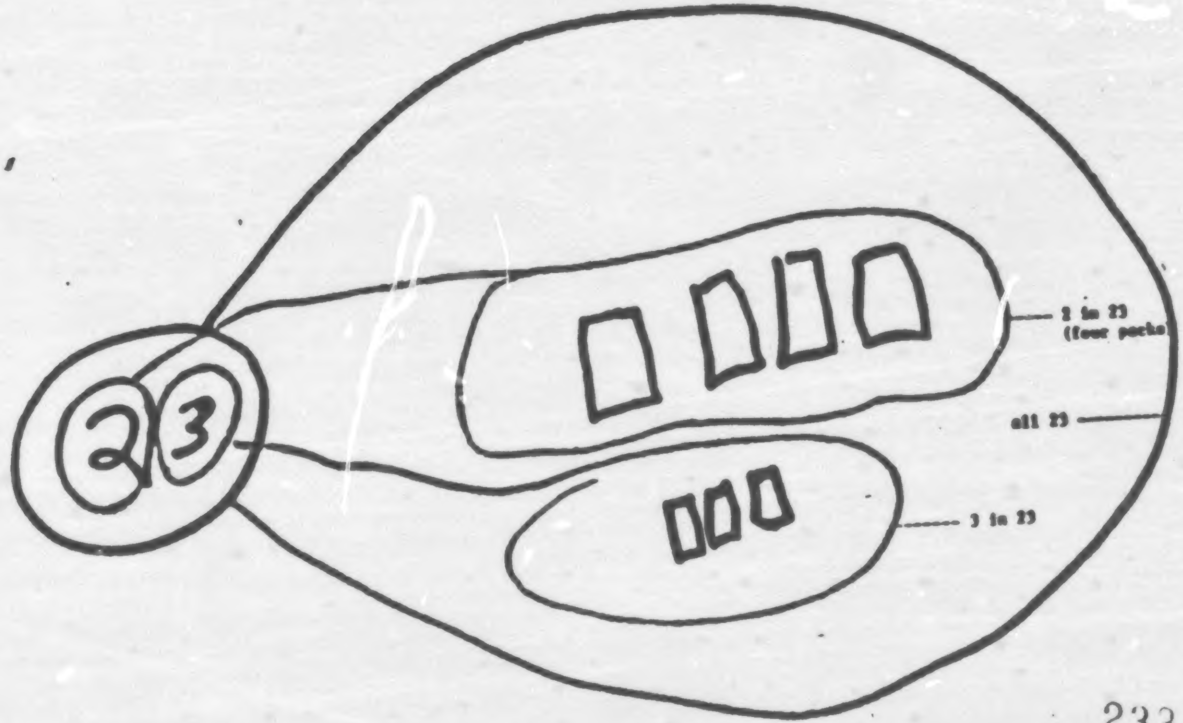
230

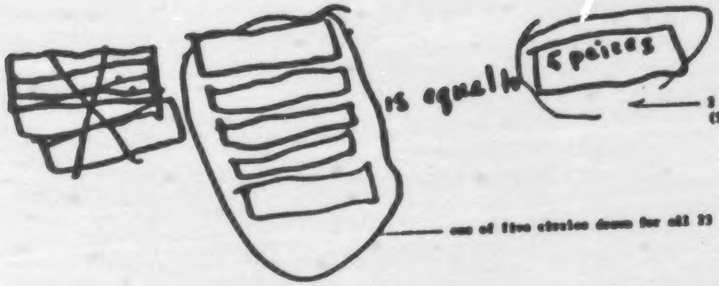

2 in 23, later changed to twenty
3 in 23
all 23
23

[illegible]

$$\begin{array}{r} 4 \\ -\ [illegible] \\ \hline \end{array}$$

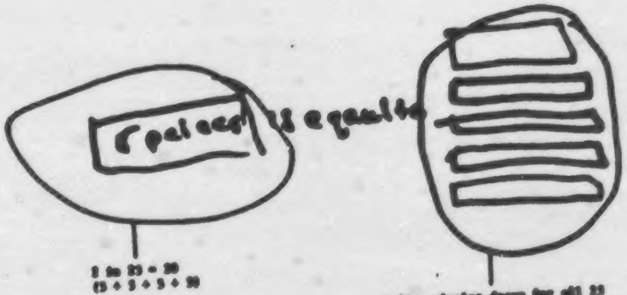
$$\begin{array}{r} 5^{pa} \\ -\ 2^{st} \\ \hline {}_{pa}4\ 3^{st} \end{array}$$


2
3
2 in 23
(four packs)
all 23
3 in 23


is equal to
5 peices

5 in 23 = 20
(5 + 5 + 5 + 5)

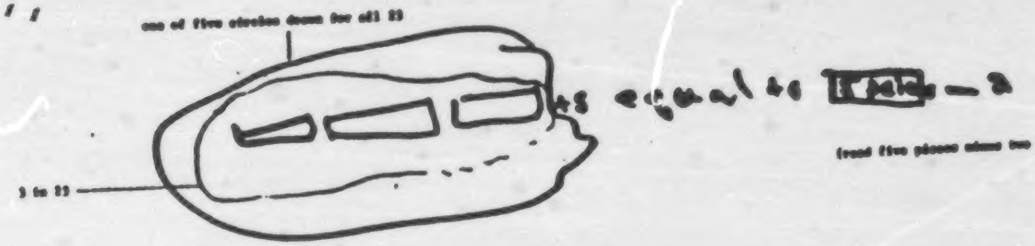
one of five circles drawn for all 23


5 peices
is equal to

5 in 23 = 20
(5 + 5 + 5 + 5)

one of five circles drawn for all 23

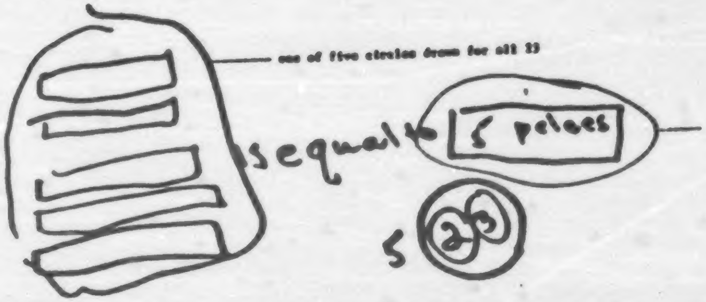
one of five circles drawn for all 23


is equal to 5 peices – 2

3 in 23

(read five pieces minus two pieces)

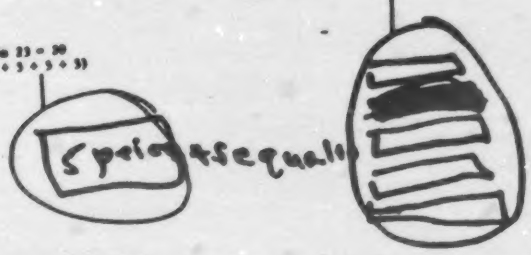
one of five circles drawn for all 23

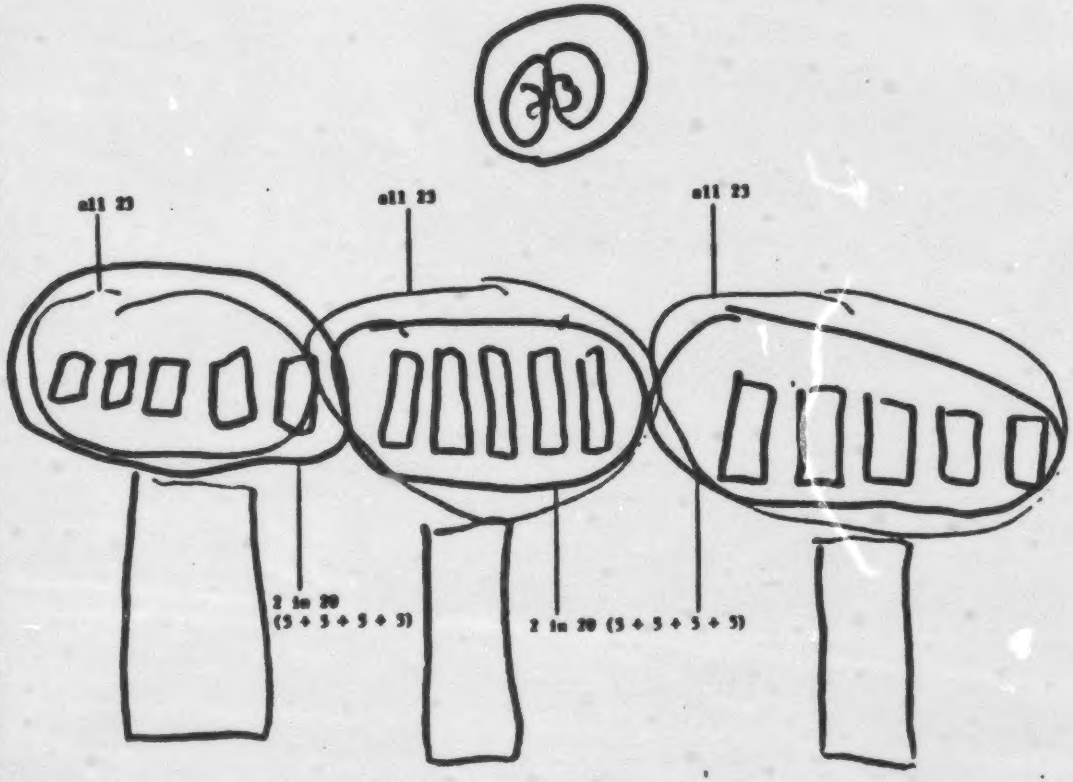

is equal to
5 peices
5
23

5 in 23 = 20
(5 + 5 + 5 + 5)

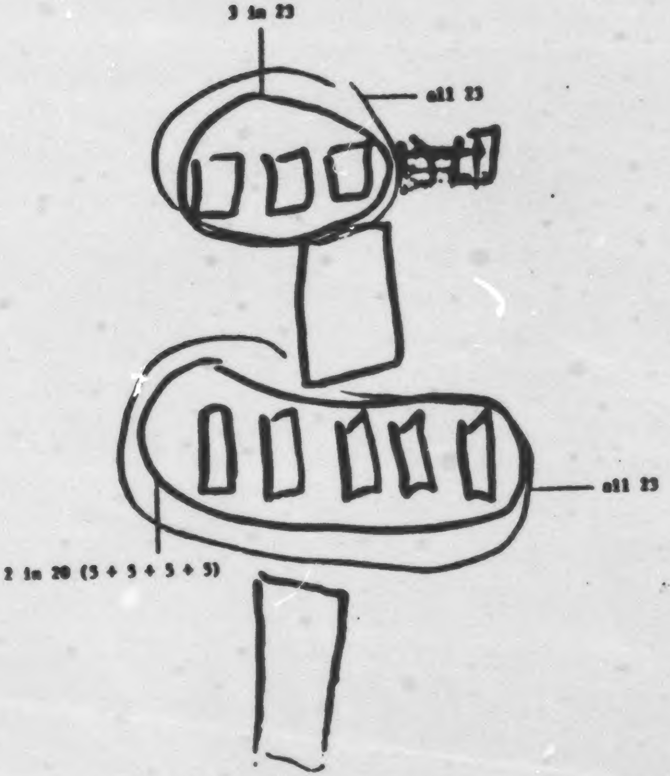
one of five circles drawn for all 23

5 in 23 = 20
(5 + 5 + 5 + 5)

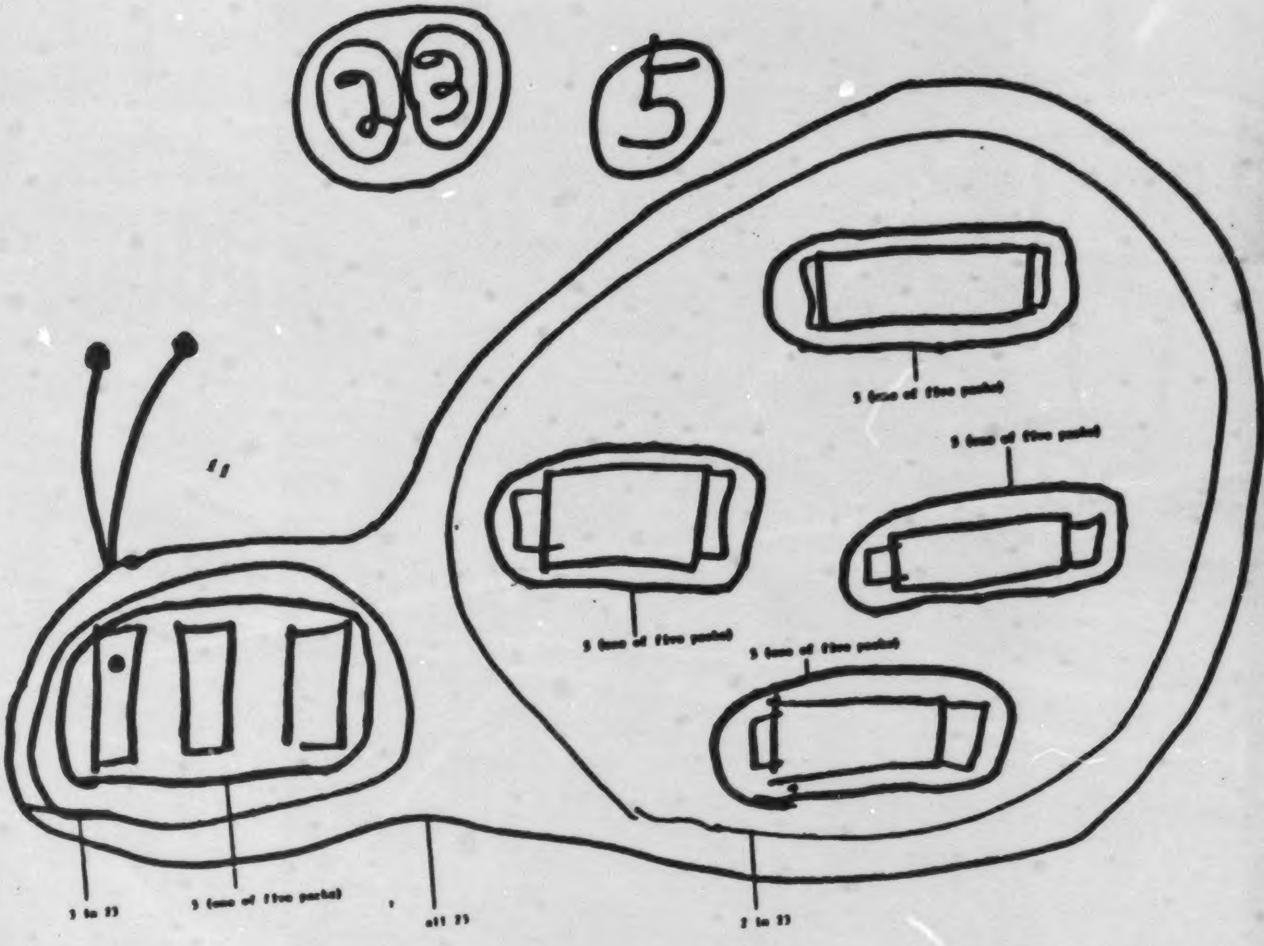

5 peices
is equal to


all 23
all 23
all 23
2 in 20
(5 + 5 + 5 + 5)
2 in 20 (5 + 5 + 5 + 5)

236

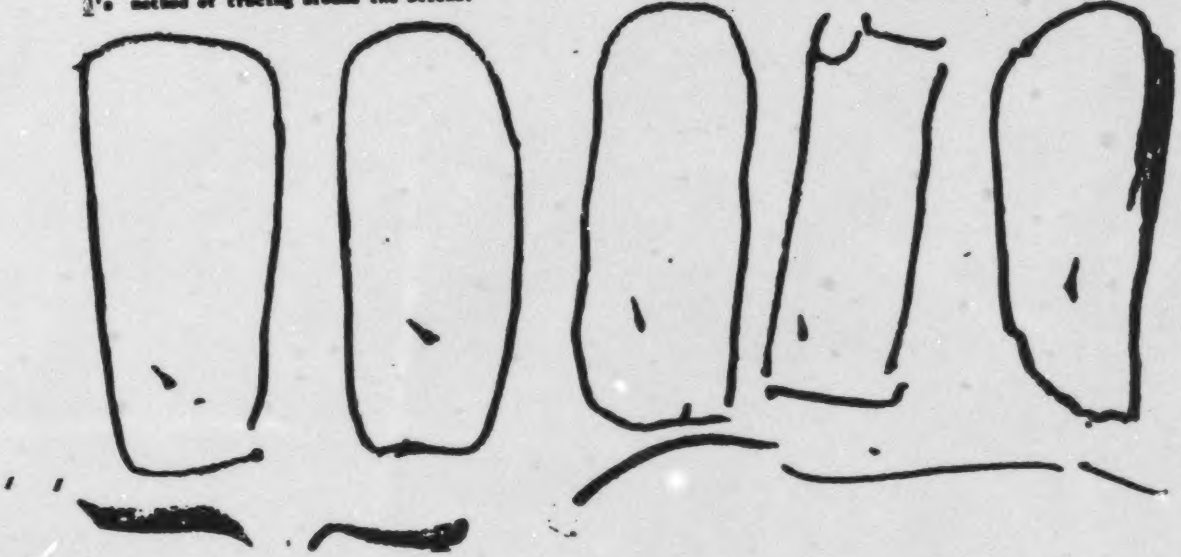

3 in 23
all 23
all 23
2 in 20 (5 + 5 + 5 + 5)

237

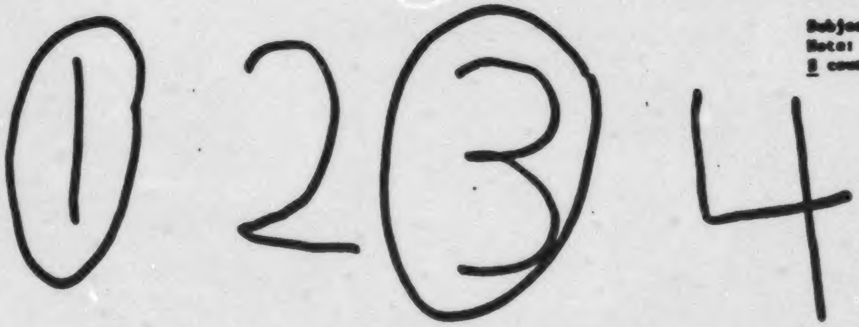

23
5
3 (one of five parts)
3 (one of five parts)
3 (one of five parts)
3 (one of five parts)
3 in 23
3 (one of five parts)
all 23
2 in 23

Interviewer drew these three sticks by imitating S's method of tracing around the sticks.

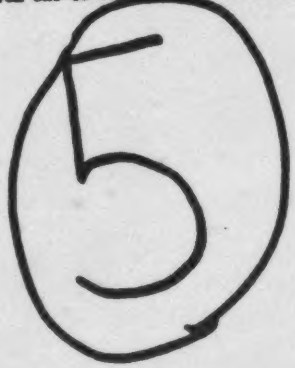
Subject drew these two sticks by drawing a line around the objects which were placed on the paper.



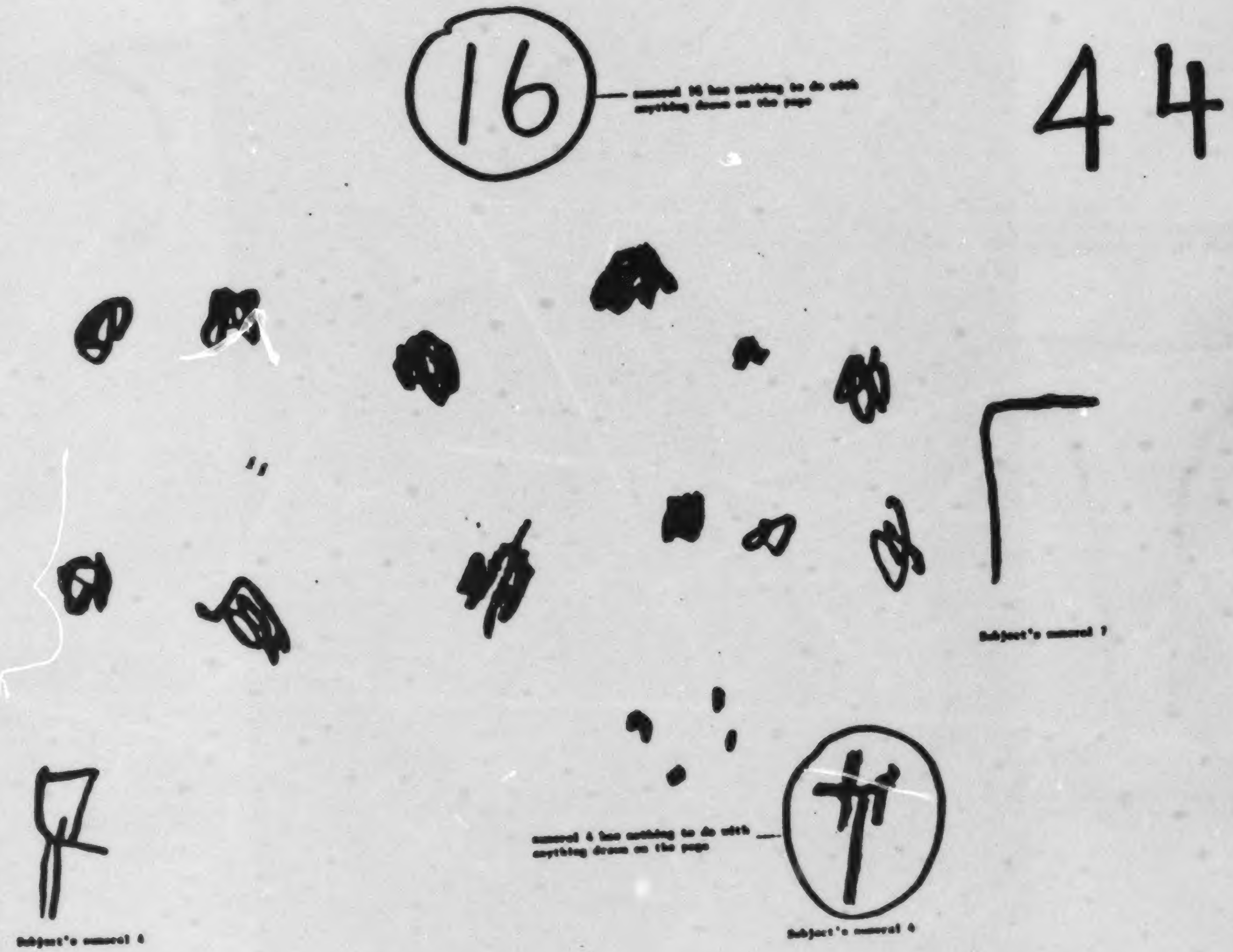
Subject's version of numerals consists of the lines underneath the sticks. Note: the S drew one line for each stick, and then recounted the sticks. S counted from one-to-five correctly each time.



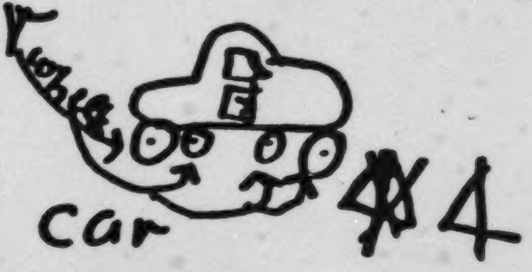
"For stuff you buy."



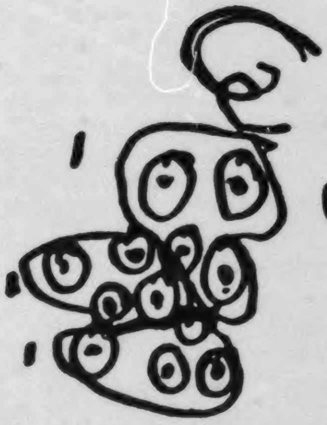
"For things you buy."

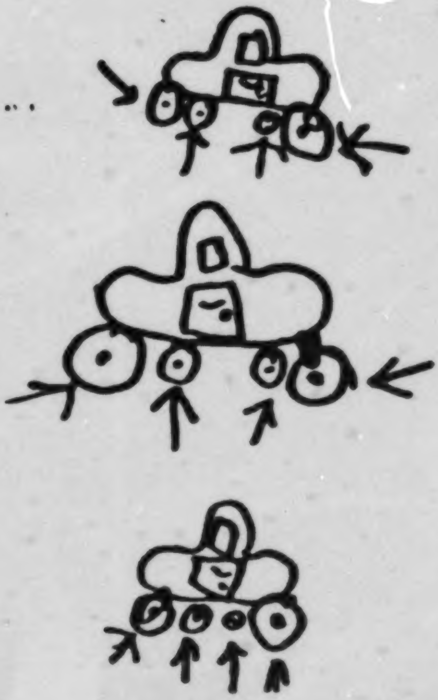

16
numeral 16 has nothing to do with anything drawn on the page
44
Subject's numeral 7
numeral 4 has nothing to do with anything drawn on the page
Subject's numeral 4
Subject's numeral 4

ther are 4 wheeles on one car


wheeles
1 car
~~4~~ 4

"Cars can be made



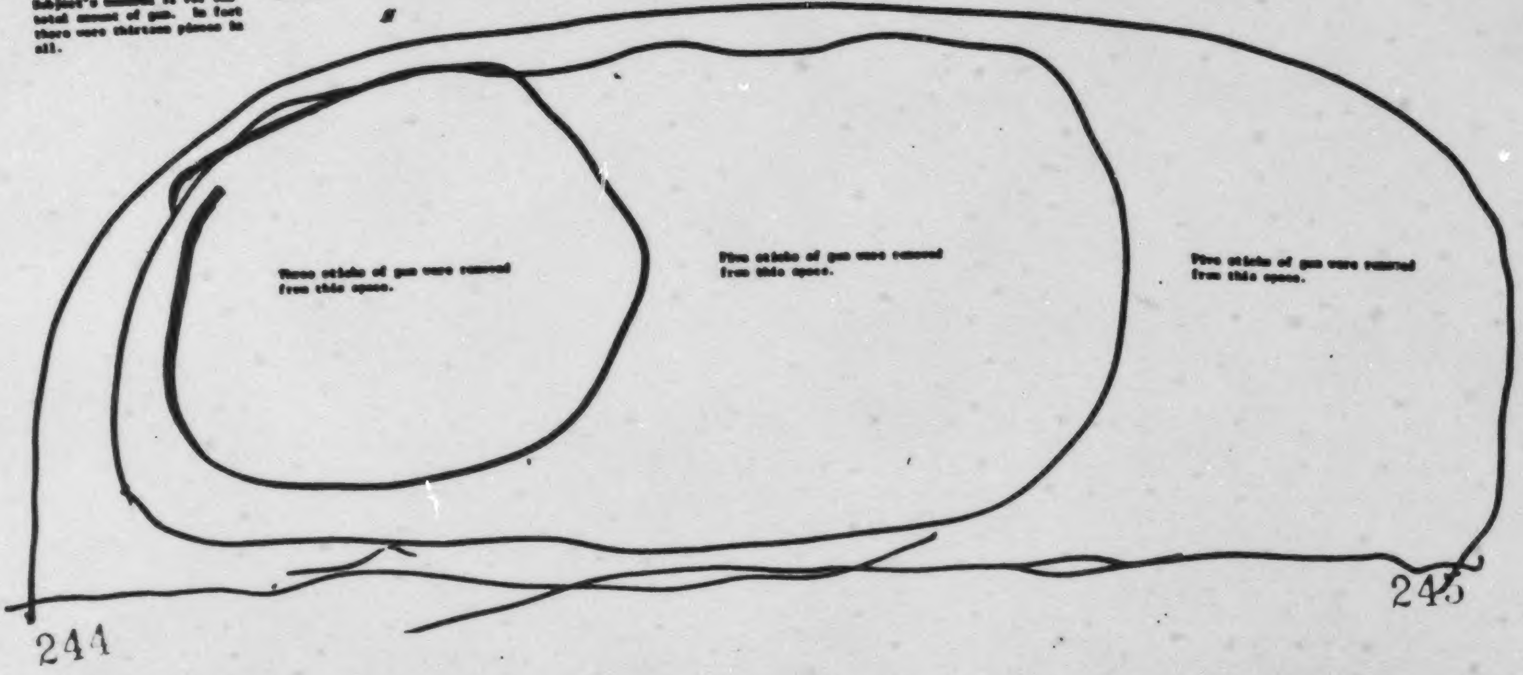


21 8 Ɛ

Subject's numeral 21 for the total amount of gum. In fact there were thirteen pieces in all.

Subject's numeral 8 (counted and written correctly)

Subject's numeral 3 (counted and written correctly)

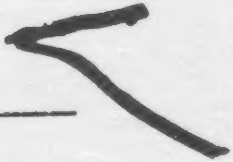


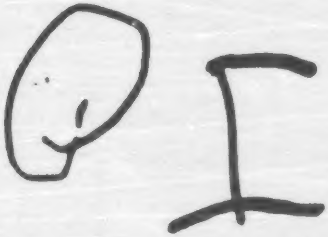
Three sticks of gum were removed from this space.

Five sticks of gum were removed from this space.

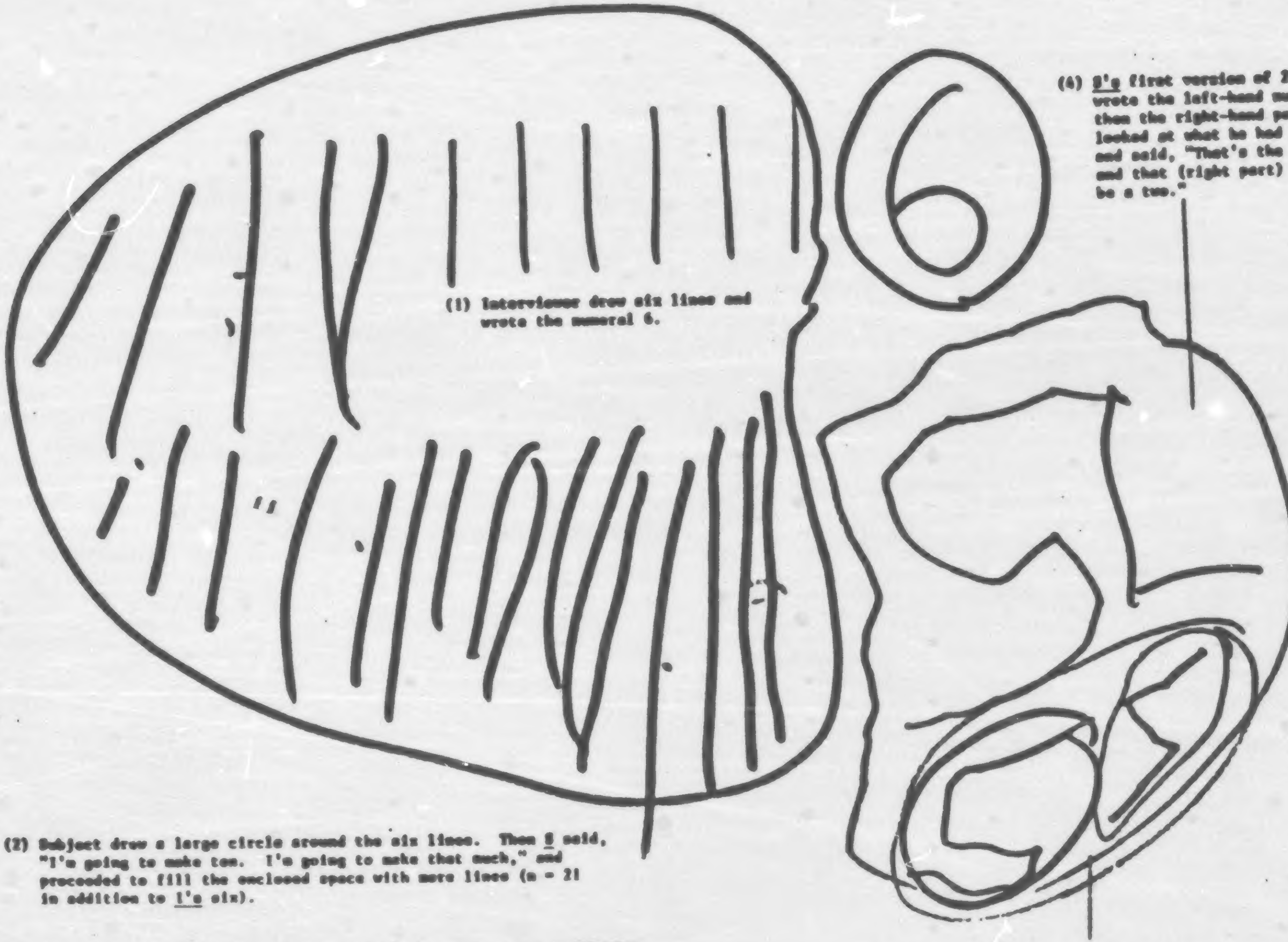
Five sticks of gum were removed from this space.

Subject's numeral 7 ————————





Subject's numeral 10. The digits are written in different colors.



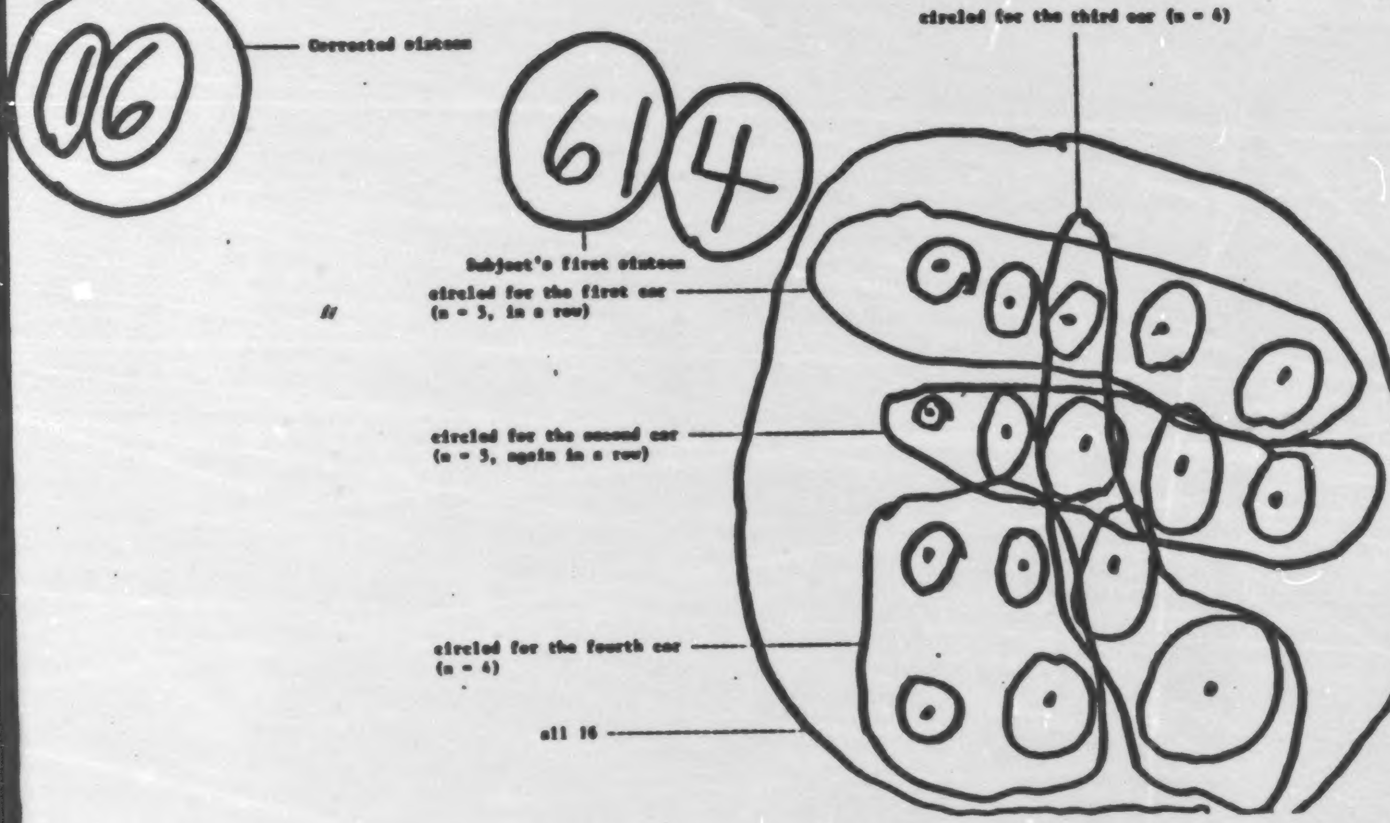
(1) Interviewer drew six lines and wrote the numeral 6.

(2) Subject drew a large circle around the six lines. Then S said, "I'm going to make ten. I'm going to make that much," and proceeded to fill the enclosed space with more lines (n = 21 in addition to I's six).

(3) S counted his drawn lines and concluded that there were twenty-two.

(4) S's first version of 22: wrote the left-hand numeral, then the right-hand portion, looked at what he had written, and said, "That's the twenty, and that (right part) should be a two."

(5) S's second attempt to write 22.
(6) S decoded the right-hand 2 as "Twenty-two brown lines."
(7) The left-hand 2 , as well as the whole numeral, were at first decoded as being twenty-two brown lines as well.
(8) S added, with reference to the line around the whole numeral, "It's a little bigger than only lines."


16
Corrected sixteen
614
Subject's first sixteen
circled for the third car (n = 4)
circled for the first car (n = 5, in a row)
circled for the second car (n = 5, again in a row)
circled for the fourth car (n = 4)
all 16

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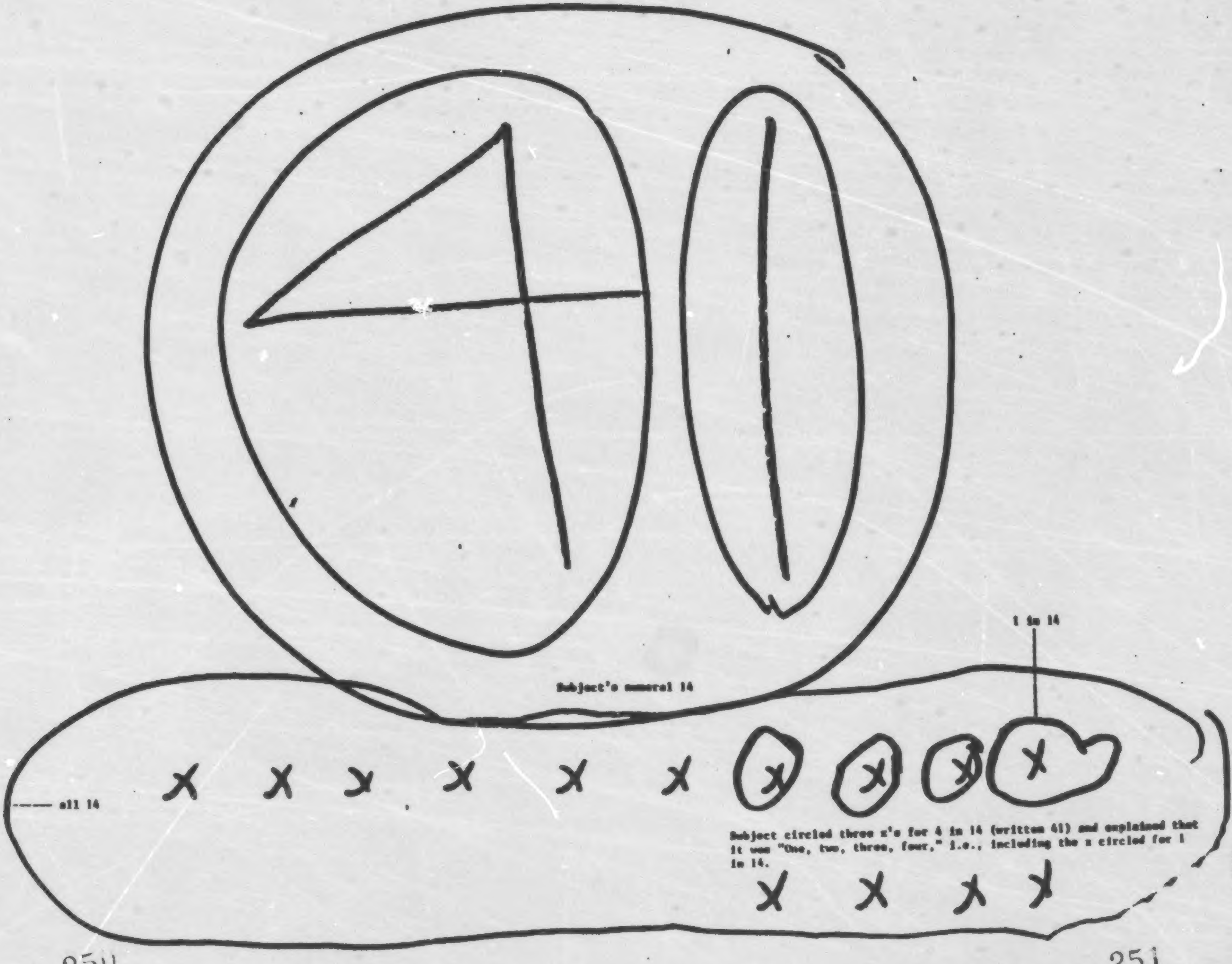
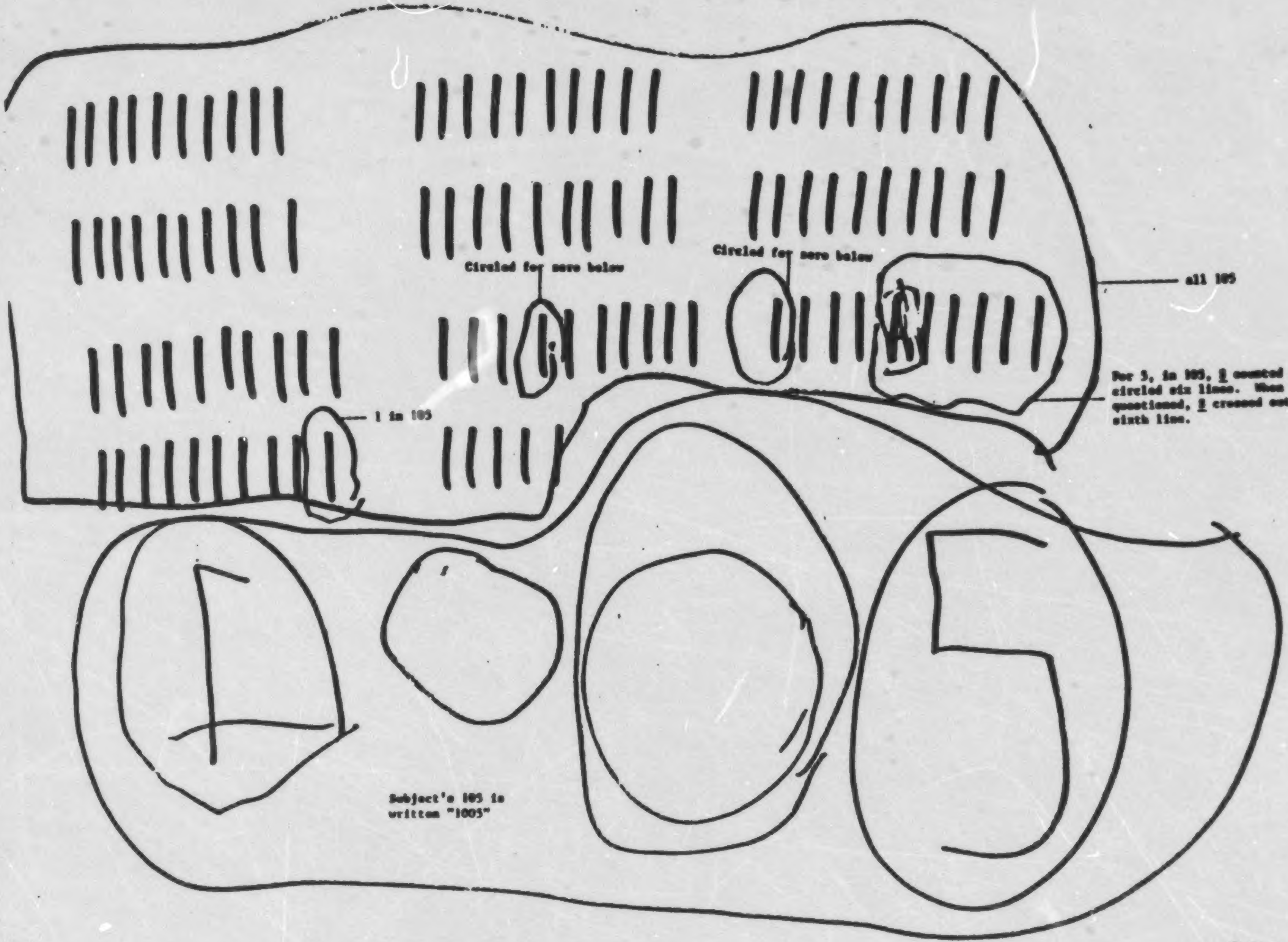

Subject's numeral 14
1 in 14
all 14

Subject circled three x's for 4 in 14 (written 41) and explained that it was "One, two, three, four," i.e., including the x circled for 1 in 14.



Circled for zero below

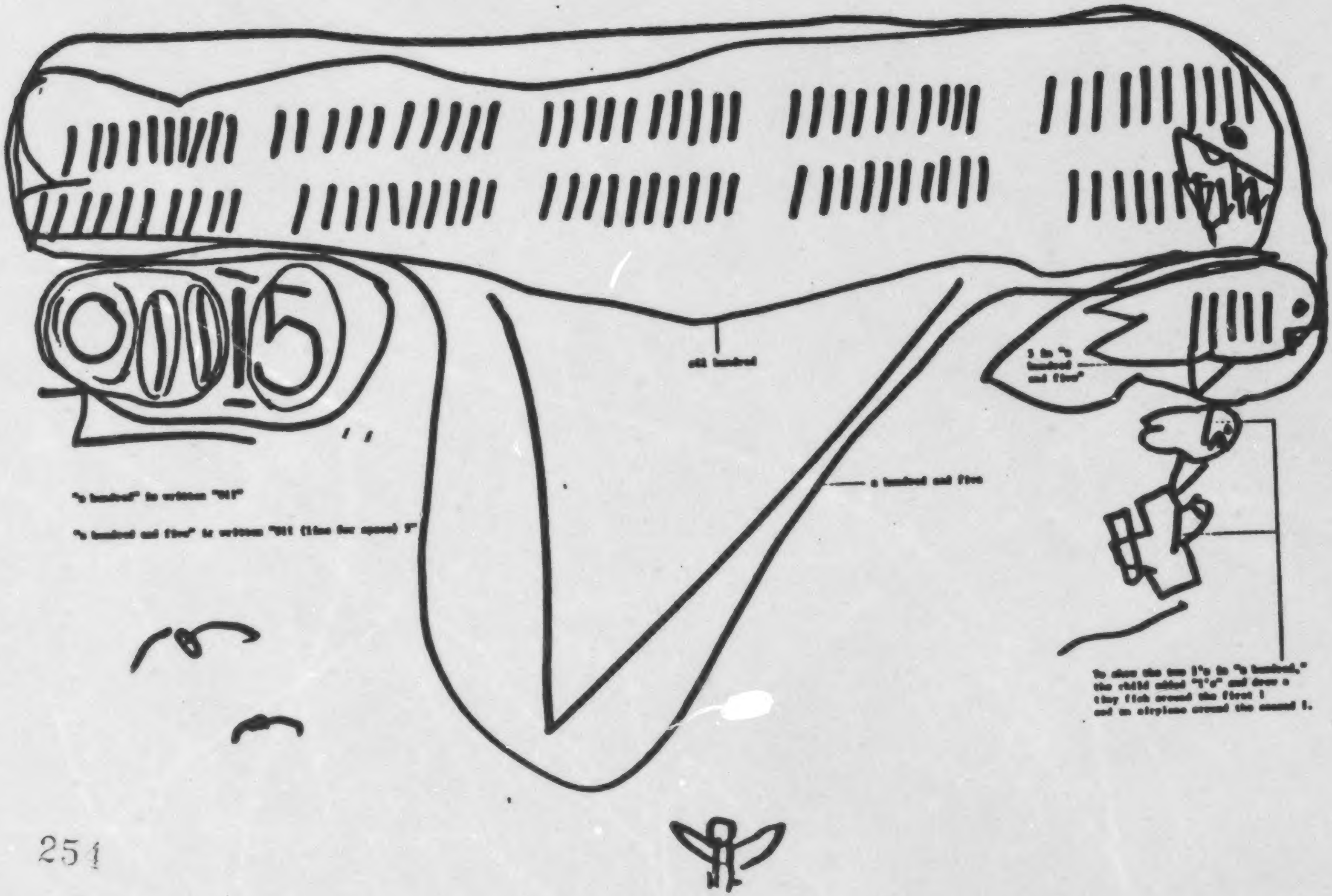
Circled for zero below

all 105

For 5, in 105, S counted and circled six lines. When questioned, S crossed out the sixth line.

1 in 105

Subject's 105 is written "1005"


"a hundred" is written "011"
"a hundred and five" is written "011 (line for space) 5"
oll hundred
a hundred and five
5 in "a hundred and five"
To show the two 1's in "a hundred," the child added "1's" and drew a tiny fish around the first 1 and an airplane around the second 1.

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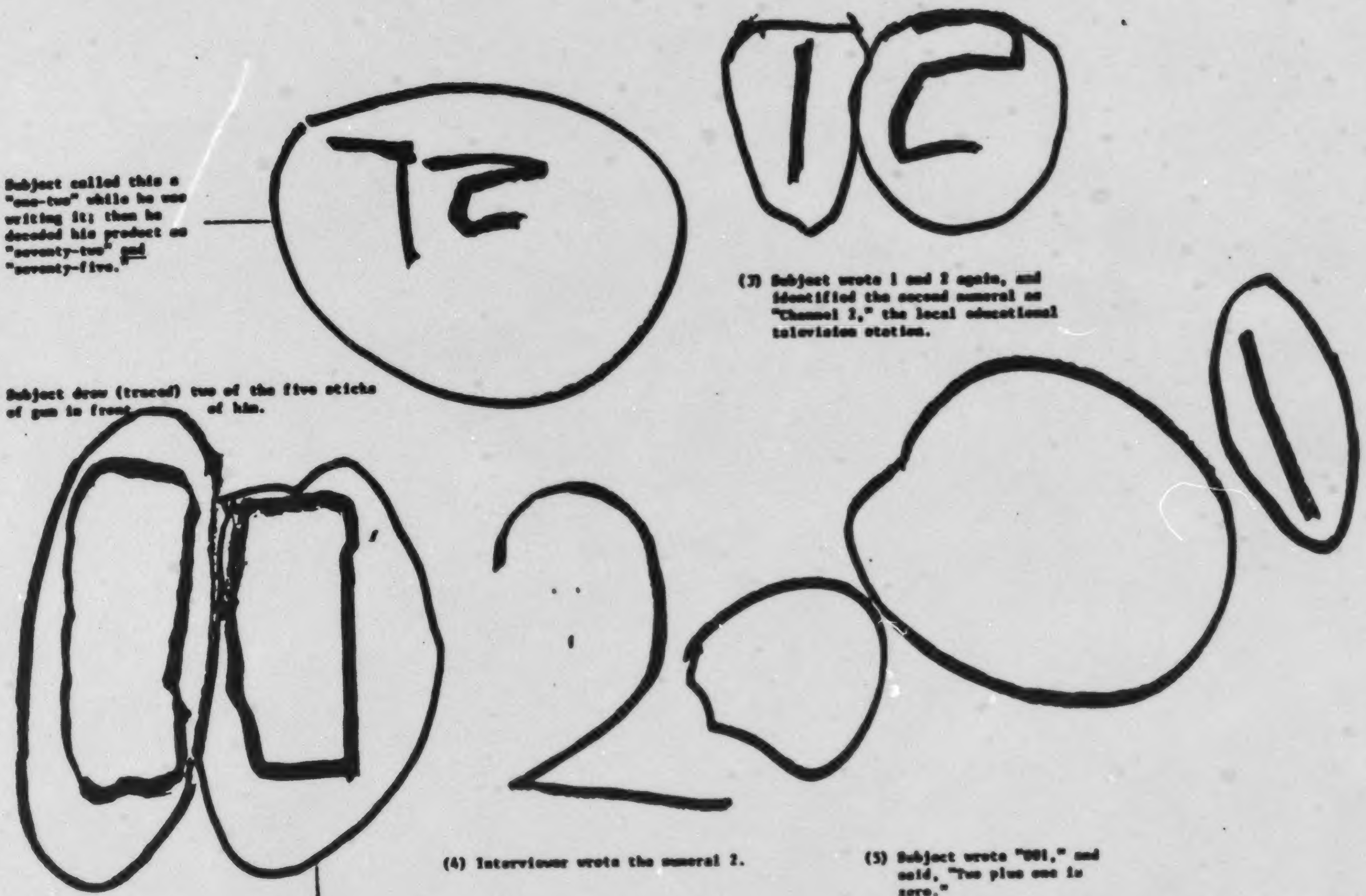

(2) Subject called this a "one-two" while he was writing it; then he decoded his product as "seventy-two" and "seventy-five."

(3) Subject wrote 1 and 2 again, and identified the second numeral as "Channel 2," the local educational television station.

(1) Subject drew (traced) two of the five sticks of gum in front of him.

(4) Interviewer wrote the numeral 2.

(5) Subject wrote "001," and said, "Two plus one is zero."

(6) Subject circled this stick after interviewer had circled his 1 in 2001 as well as the stick to the left of it.

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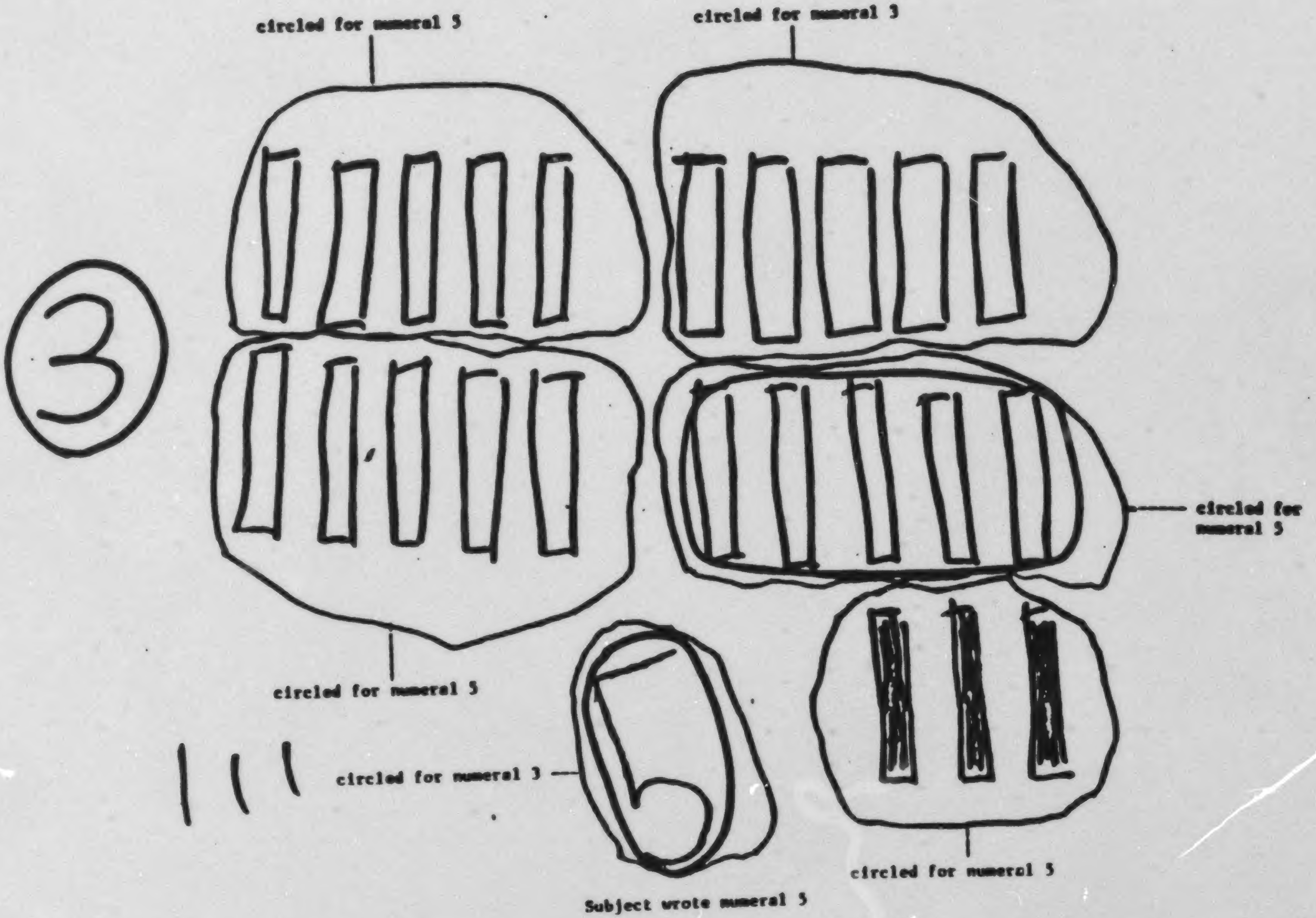
circled for numeral 3
circled for numeral 3
3
circled for numeral 3
circled for numeral 3
circled for numeral 3
circled for numeral 3
circled for numeral 3
Subject wrote numeral 3

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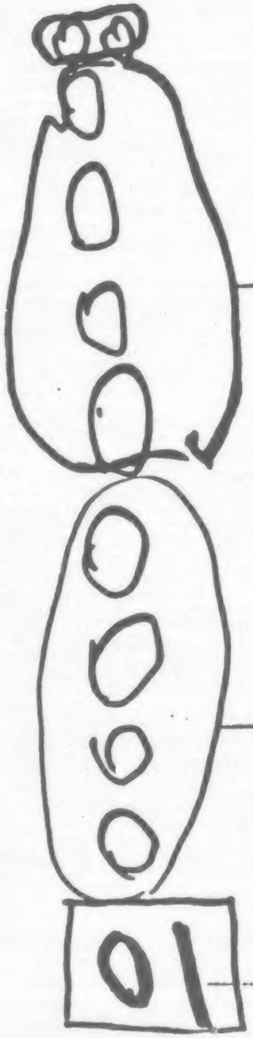
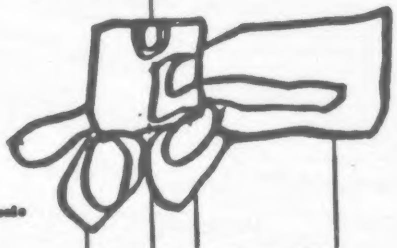


Subject drew circle around "wheels for another car"

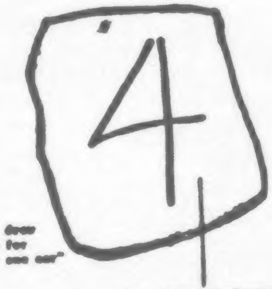
Interviewer drew this circle for "wheels for one car"

Subject wrote the numeral (10) before drawing the ten wheels

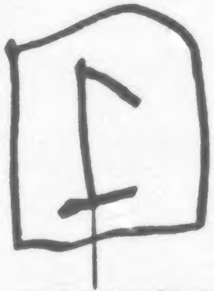


Subject decoded numeral 1 as one part of his car

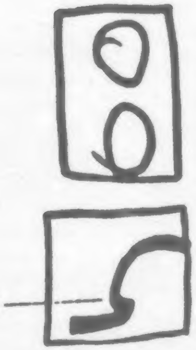
Subject later decoded numeral 4 as "one, two, three, four (parts of his car)"



Subject decoded the numeral as "four years old, so I'm four."



Numeral decoded as "one wheel and one car"



Subject wrote the numeral (2) before drawing the two wheels

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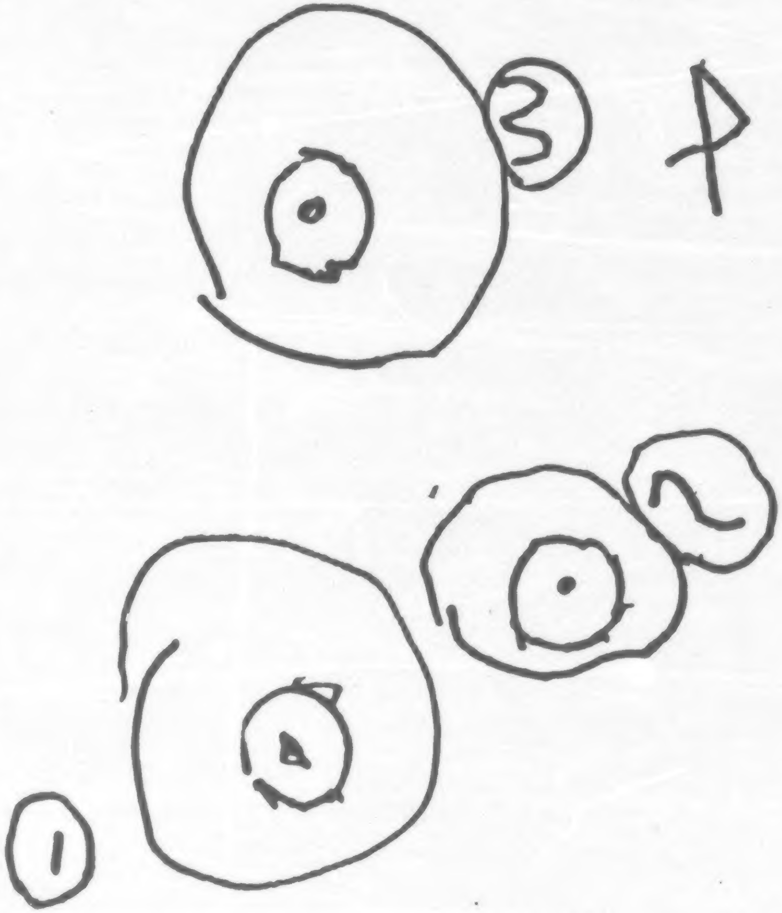
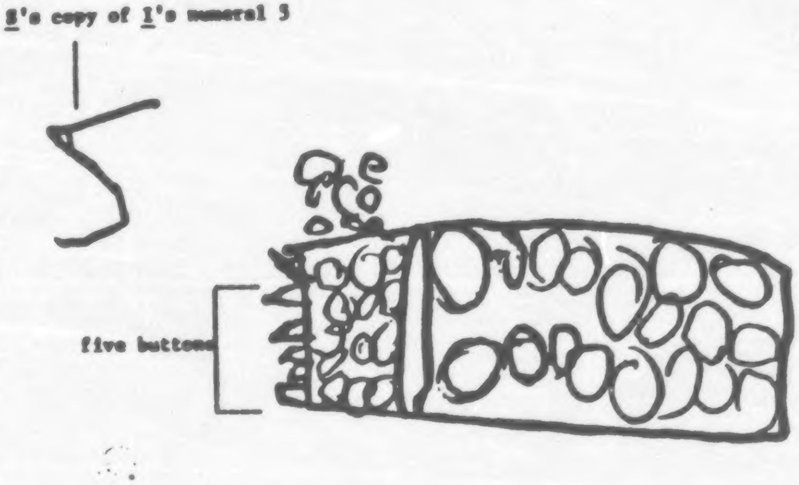


(1) Subject traced around three wheels.
(2) S labeled them "one wheel, two wheel, three," with numerals.
(3) S wrote the numeral 4: "That's how many there was before. Four. Four years old, before I was five."
(4) S drew circles around "1" and "2" in response to Interviewer's circling of "3."


S's copy of I's numeral 5
five buttons

Drawing of the tape recorder. S drew six buttons, then counted the buttons on the recorder (n = 5), and crossed out the sixth button.

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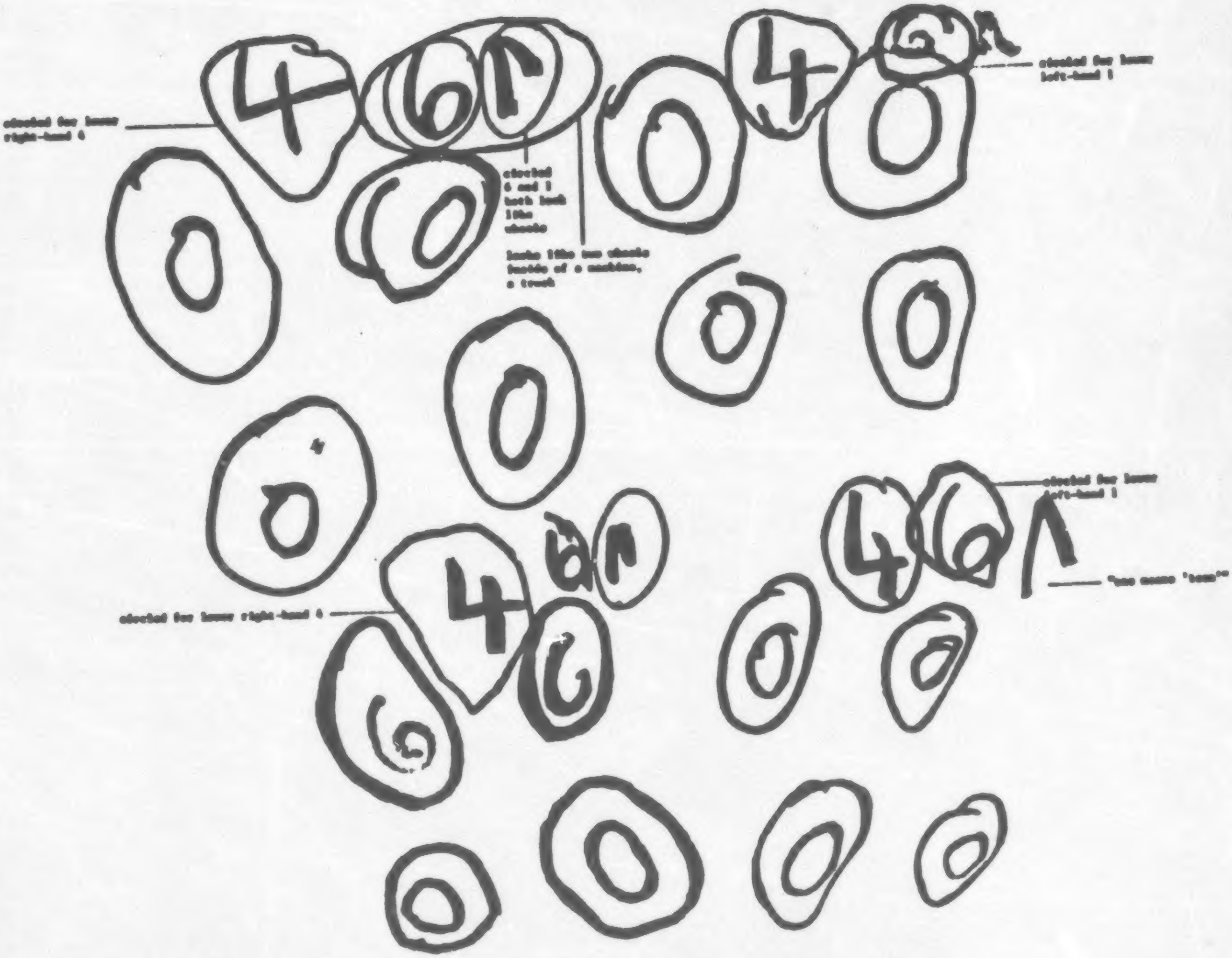
circled for lower right-hand 6
circled 6 and 1 both look like wheels
looks like two wheels inside of a machine, a truck
circled for lower left-hand 1
circled for lower right-hand 6
circled for lower left-hand 1
"one means 'turn'"

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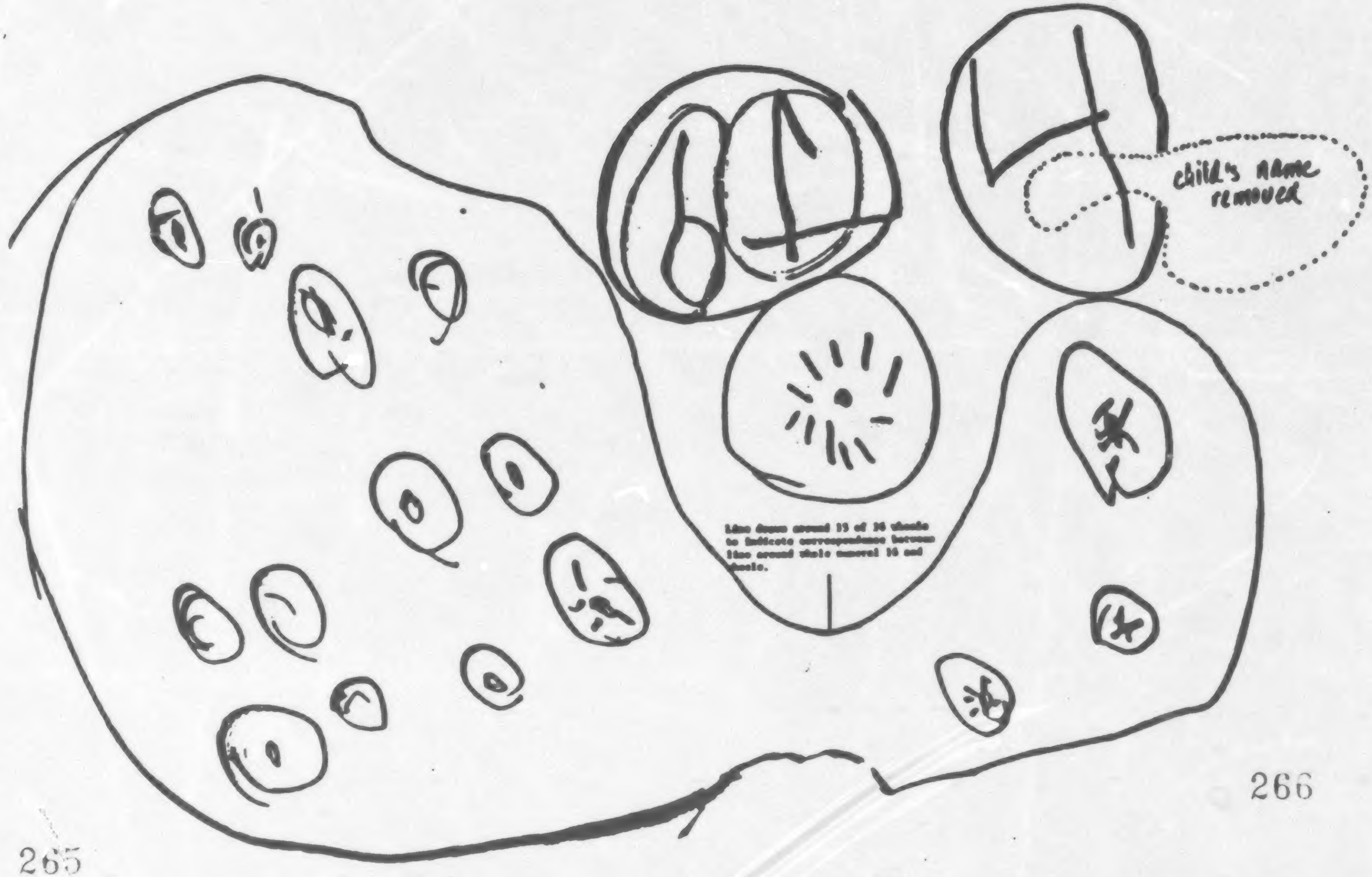
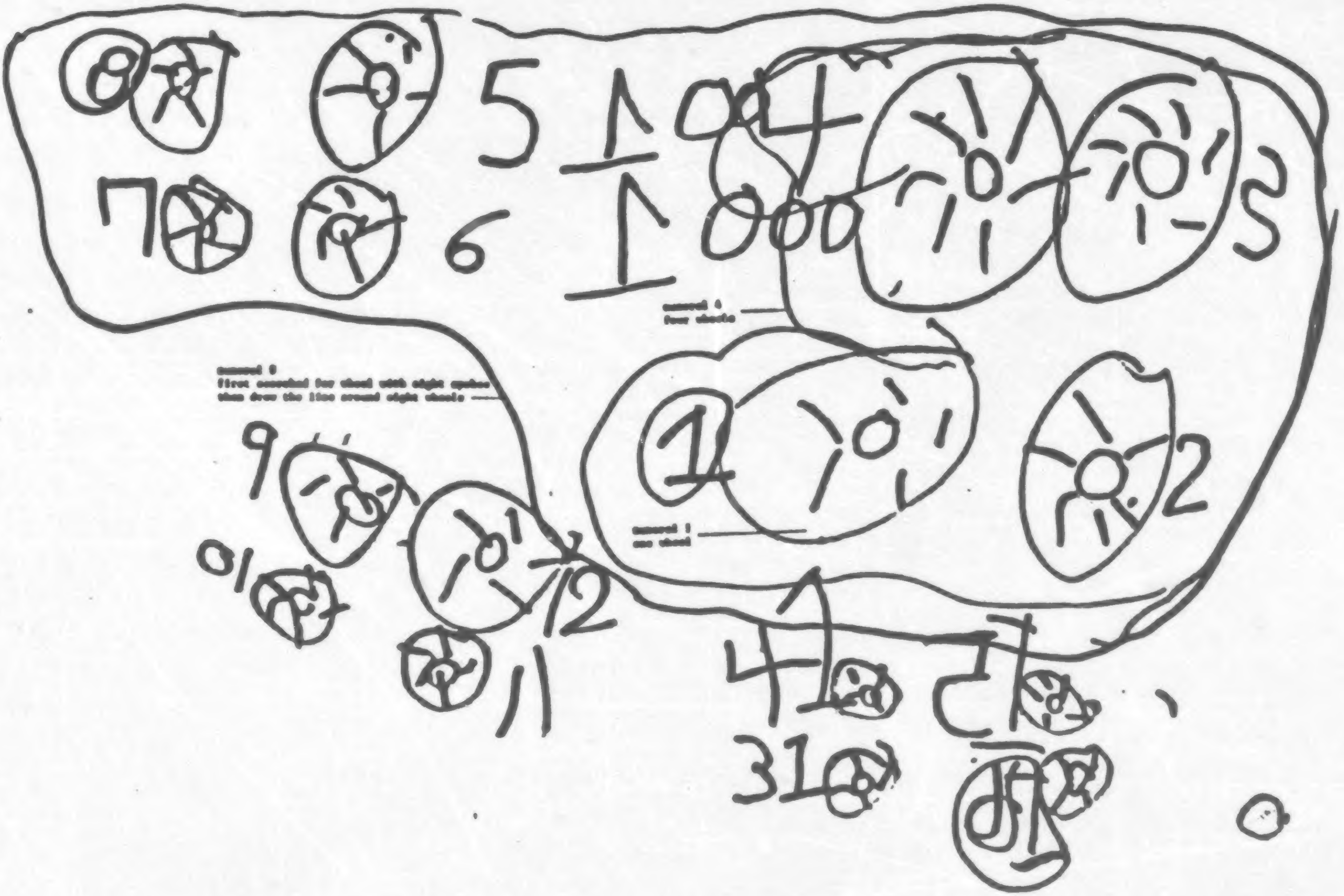


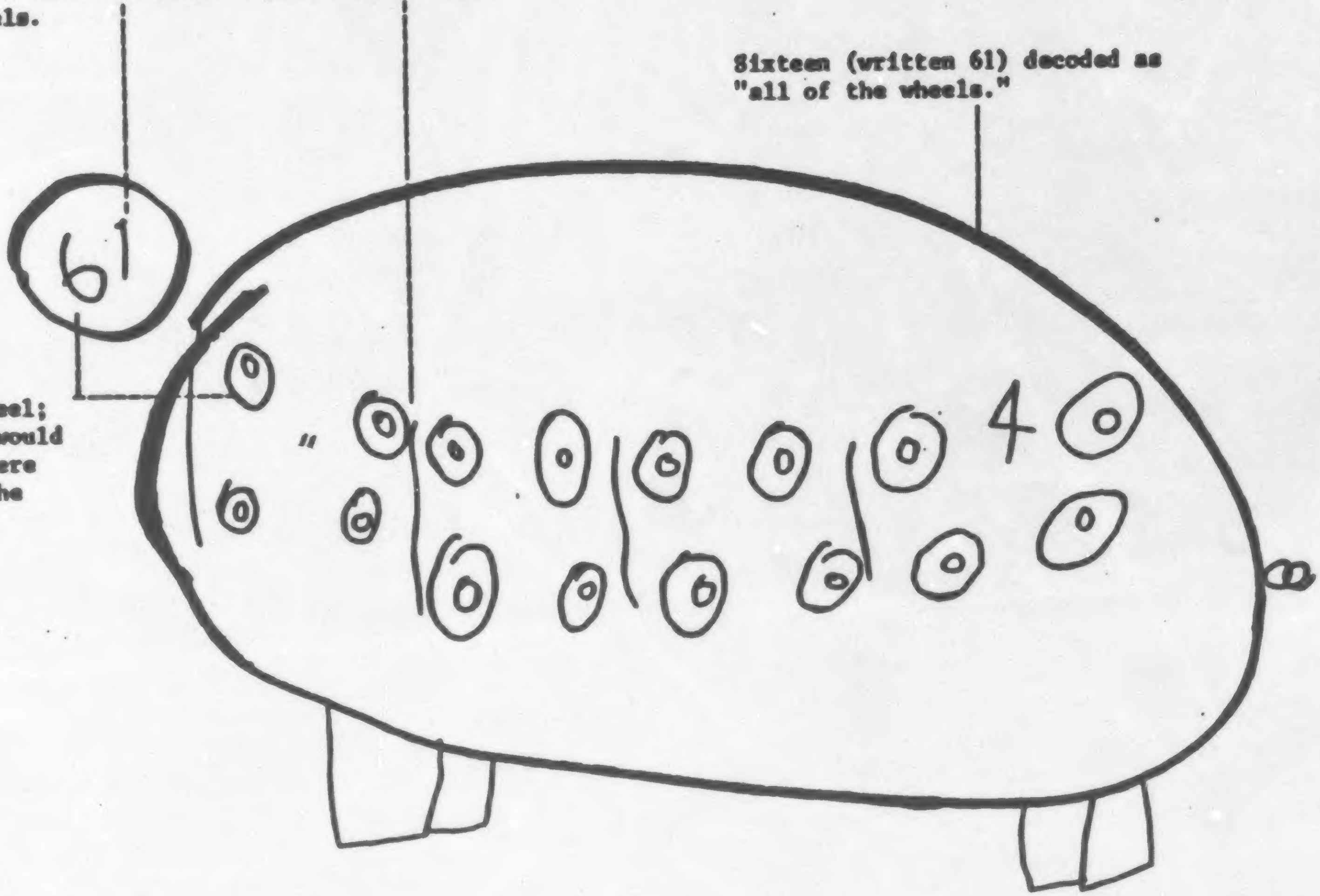
Line drawn around 15 of 16 circles to indicate correspondence between line around whole numeral 16 and circles.

child's name removed

265

266

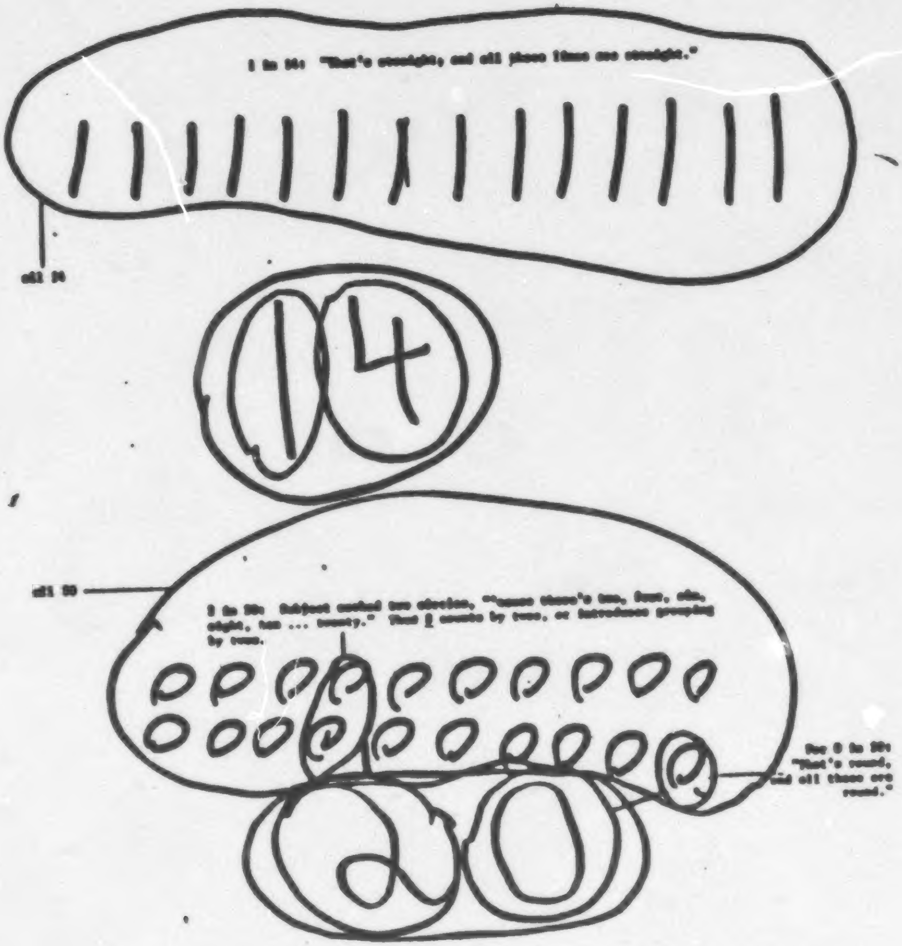

5 1 00 4
7 6 1 000 5
numeral 4
four wheels
numeral 8
first searched for wheel with eight spokes
then drew the line around eight wheels
1 2
numeral 1
one wheel
9
01
12
11
41 5
31



Without the hook (at the top of the numeral 1), it would look like the lines separating the sets of four wheels.

Sixteen (written 61) decoded as "all of the wheels."

The 6 in 16 looks like a wheel; correspondence would be exact "if there was a hole in the middle."



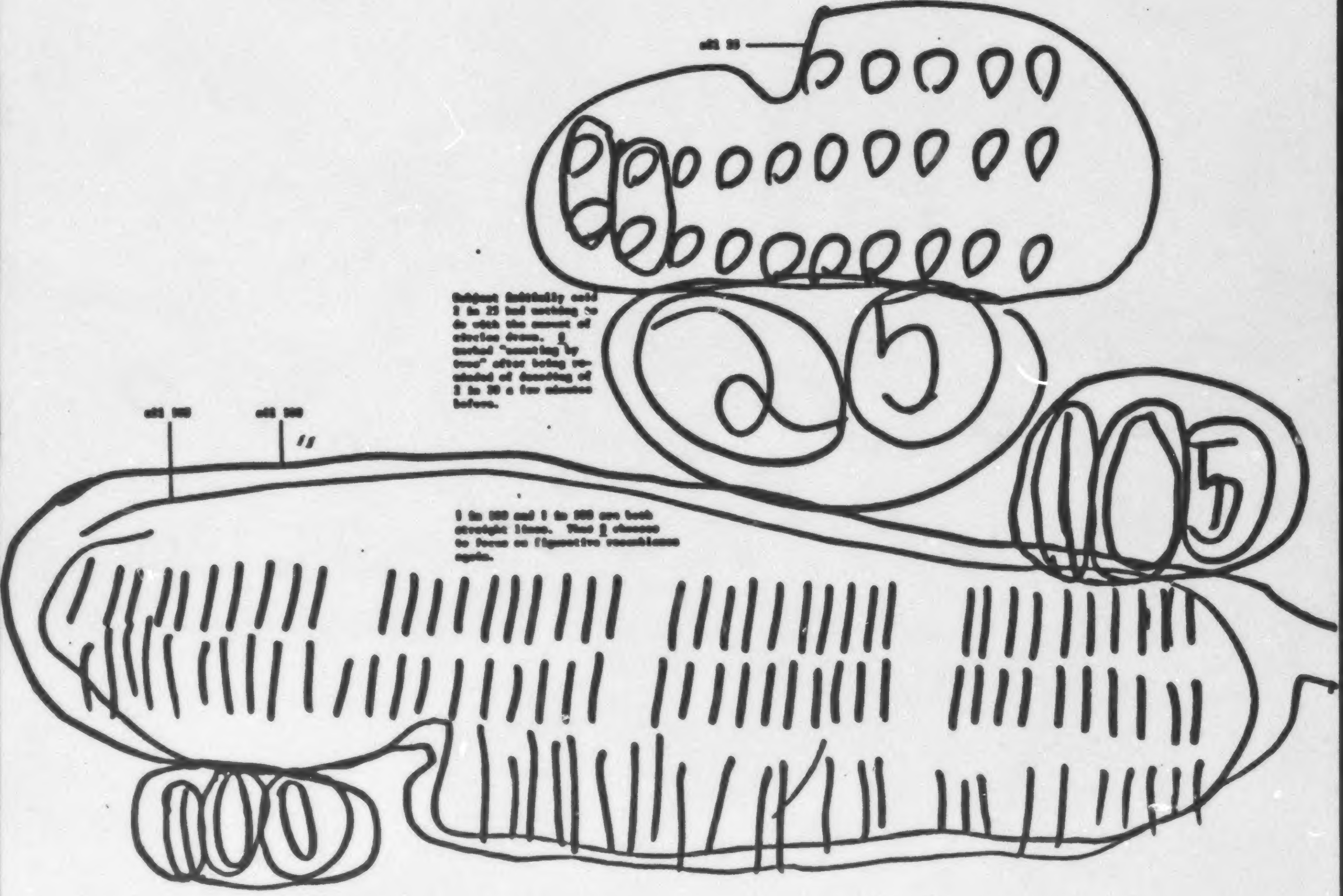
I in 14: "That's straight, and all these lines are straight."

all 14

all 20

I in 20: Subject marked two circles, "'cause there's two, four, six, eight, ten ... twenty." Then I counts by twos, or introduces grouping by twos.

[illegible] 2 in 20: "That's round, and all these are round."



Subject initially said 1 in 25 had nothing to do with the amount of circles drawn. S started "counting by twos" after being reminded of counting of 1 in 20 a few minutes before.

1 in 200 and 1 in 300 are both straight lines. Thus S chooses to focus on figurative resemblance again.

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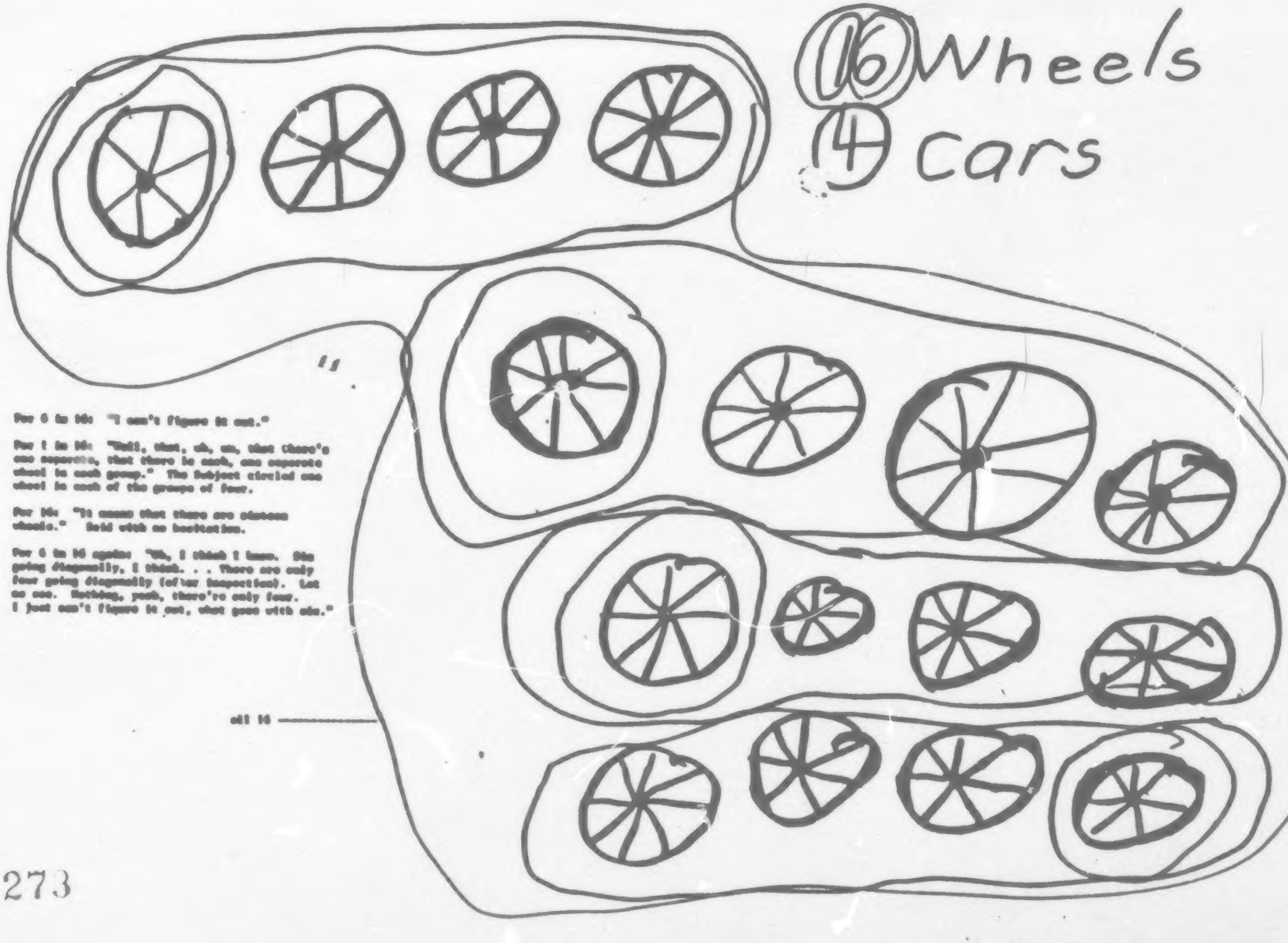


For 6 in 16: "I can't figure it out."

For 1 in 16: "Well, that, uh, um, that there's one separate, that there is each, one separate wheel in each group." The Subject circled one wheel in each of the groups of four.

For 16: "It means that there are sixteen wheels." Said with no hesitation.

For 6 in 16 again: "Oh, I think I know. Six going diagonally, I think. . . There are only four going diagonally (after inspection). Let me see. Nothing, yeah, there're only four. I just can't figure it out, what goes with six."

all 16 ——

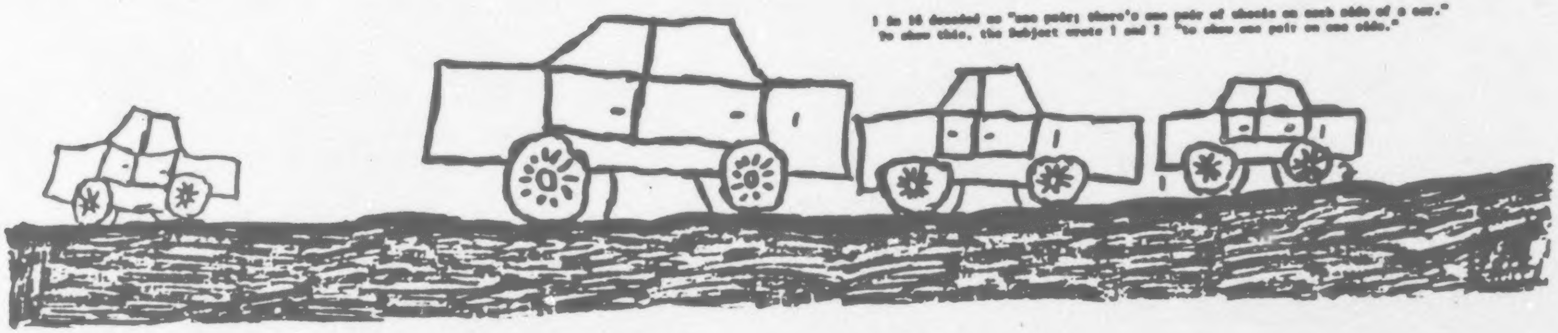
16

4

For whole 16: "there's sixteen wheels on each and every car," by which the Subject meant sixteen wheels in all.

6 in 16 decoded as "a car and a half; four wheels and two wheels," and as "one, one, one, one pair of two, two pairs of two, three pairs of two, so that's two, four, six."

1 in 16 decoded as "one pair; there's one pair of wheels on each side of a car." To show this, the Subject wrote 1 and 2 "to show one pair on one side."



sixteen
⑯
four cars
④ cars

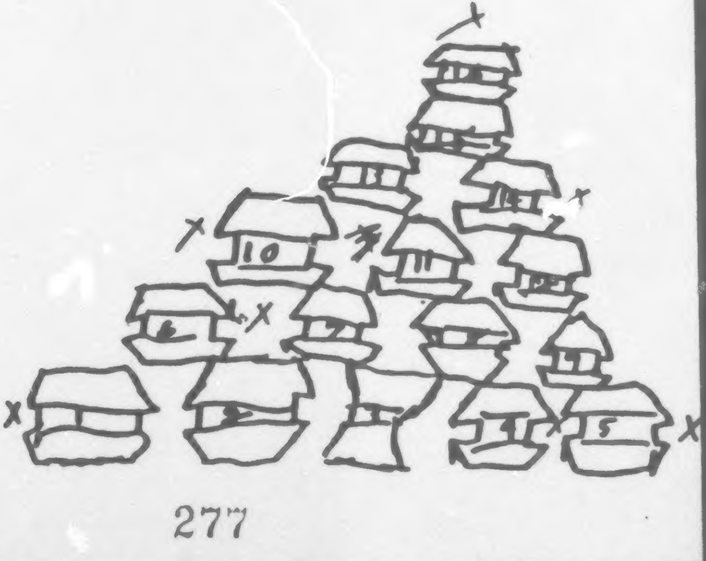
(1) For 6 in 16: Subject at first says it has nothing to do with the amount of wheels.

(2) For 1 in 16: "Could be ten," and writes the numerals 1 through 10 inside the wheels.

(3) For all 16: S̲ writes the numerals 11 through 16 in the remaining wheels.

/ / (4) When asked about the 6 in 16 again, S̲ responded, "Could be ten plus six."

(5) For 4 (cars), S̲ marks off groups of four wheels with X's.



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23
twenty-three
four packs and three

②0

②3

5

5

5

5

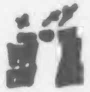
(1) Subject drew four small rectangles for four packs of gum, and three smaller rectangles for three sticks.

(2) For 3 in 23, colored in the three sticks.

(3) For 2 in 23, nothing.

(4) For all 23, colored in the four packs and placed three x's above the sticks to show "all of them."

Interviewer's first attempt to point out inconsistencies between responses on Task 5, Version 3 and Task 7 (where S immediately decoded the 2 in 20 and 2 in 23 as being "two tens," and "twenty") met with no success.



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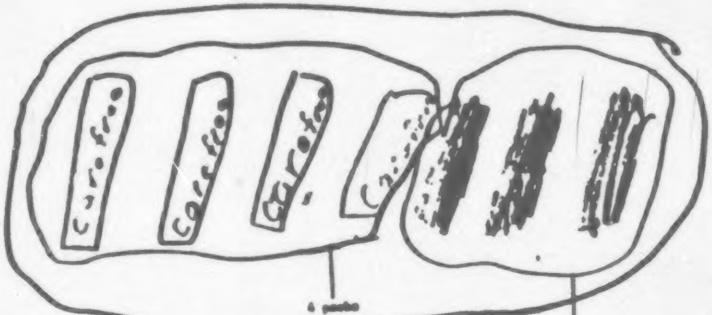

Carefree
Carefree
Carefree
Carefree
4 packs
3 in 23: three sticks
all 23

Packs
~~3~~ ① ~~Sticks~~ of gum
㉓ Sticks

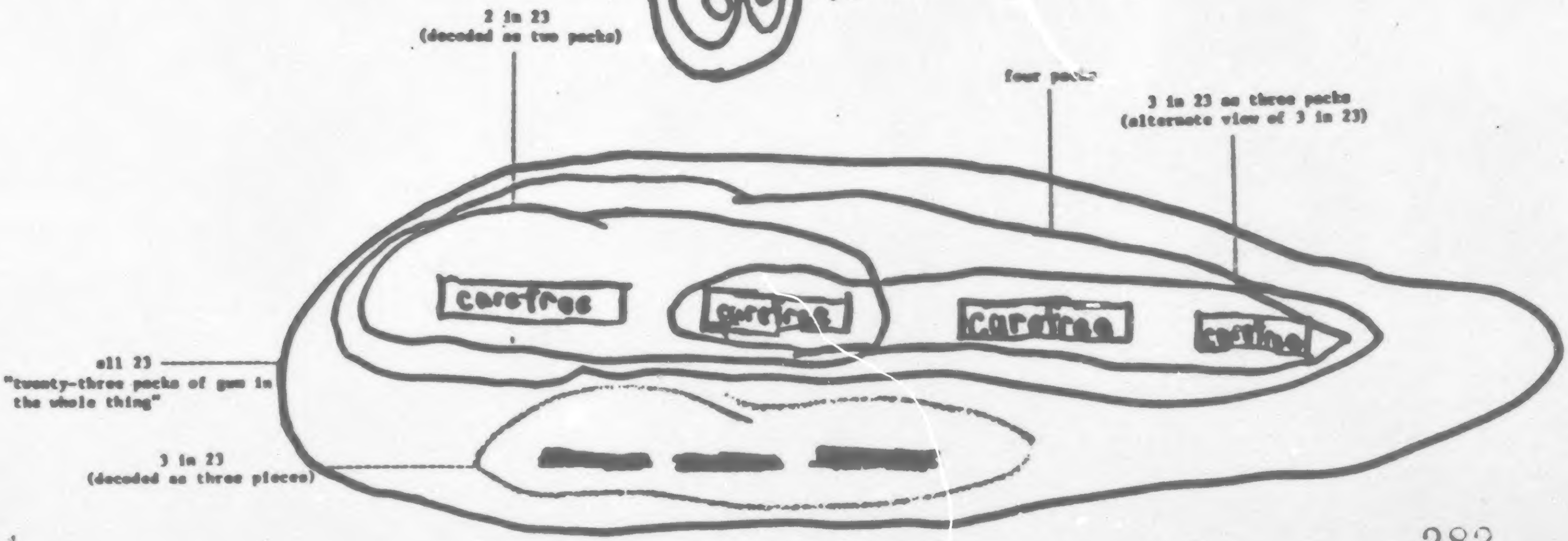
For 2 in 23: "Um, there are two kind of, like, there are two kinds of gum, like there's sticks, and the four packs. Separate. There are two kinds, you know, here," and Subject points to drawn packs and sticks.

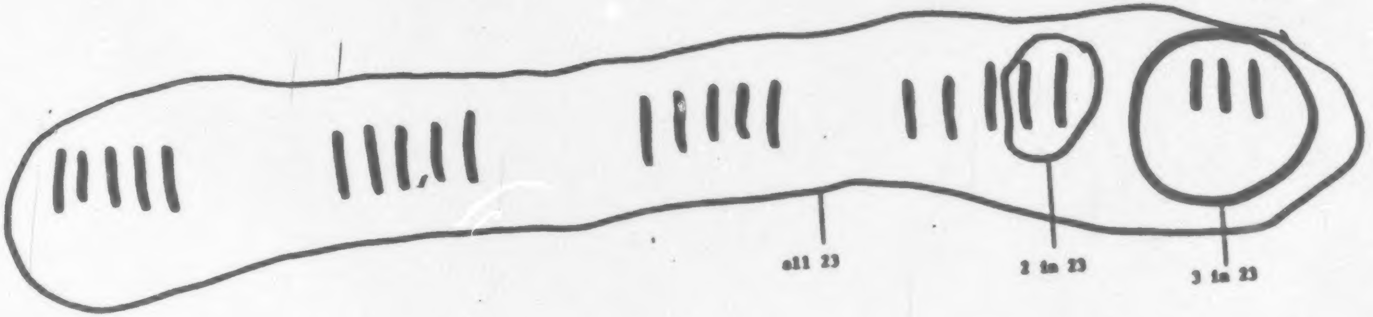
2 kinds

Sticks + Packs

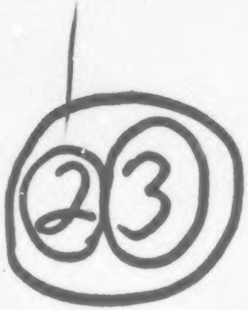
4 packs. 3 pices.

23 in all.

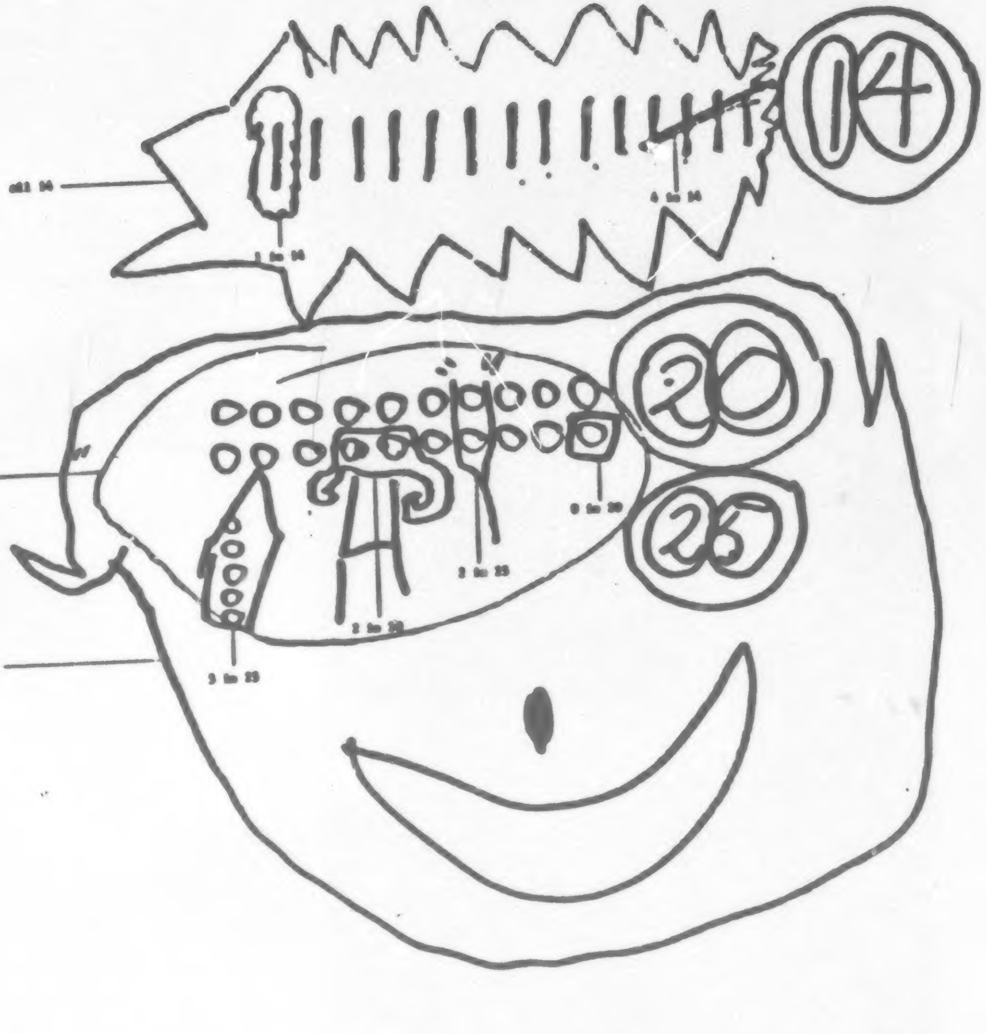

2 in 23
(decoded as two packs)
four packs
3 in 23 as three packs
(alternate view of 3 in 23)
carefree
carefree
carefree
carefree
all 23
"twenty-three packs of gum in the whole thing"
3 in 23
(decoded as three pieces)

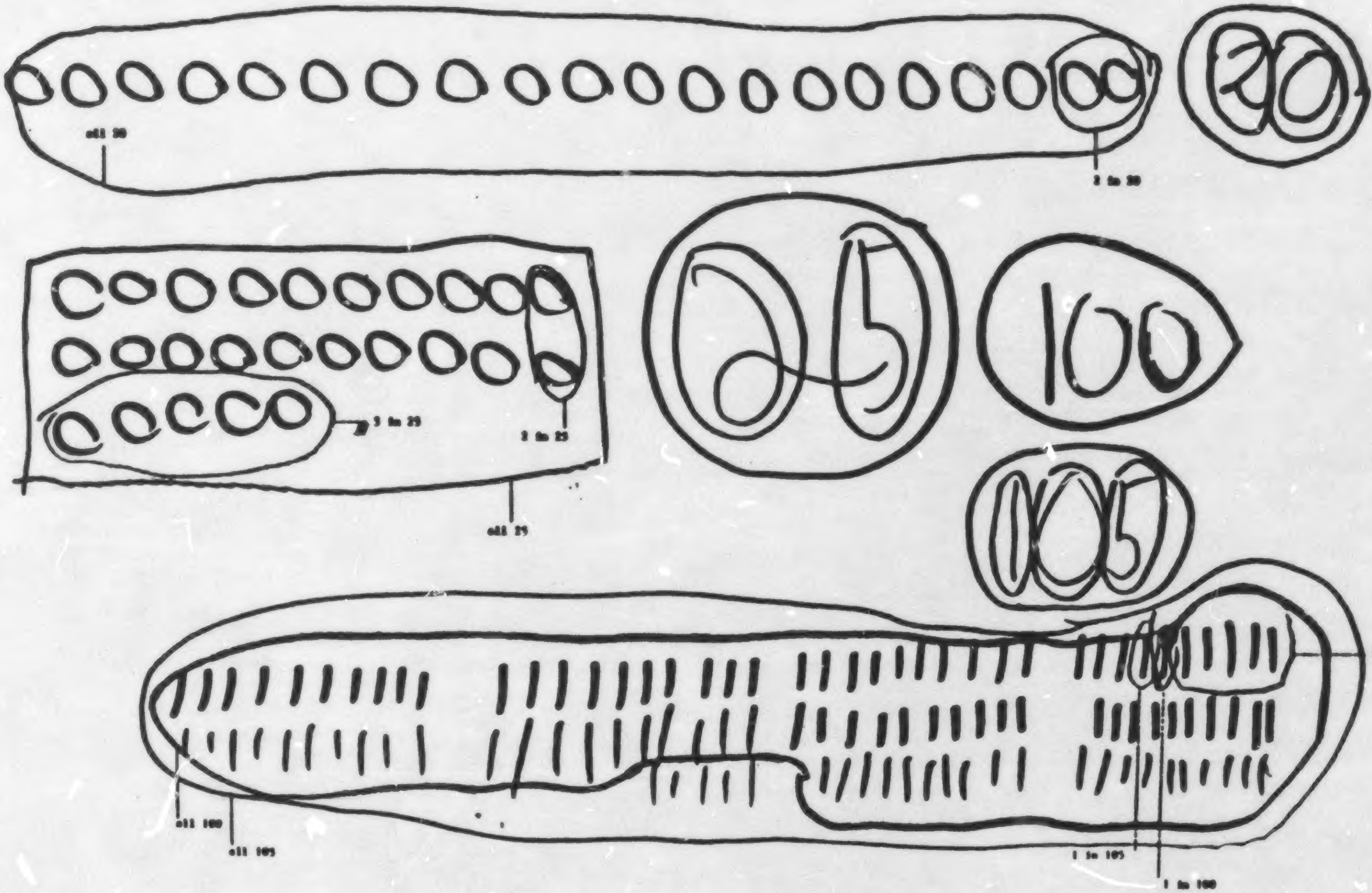

all 23
2 in 23
3 in 23

283



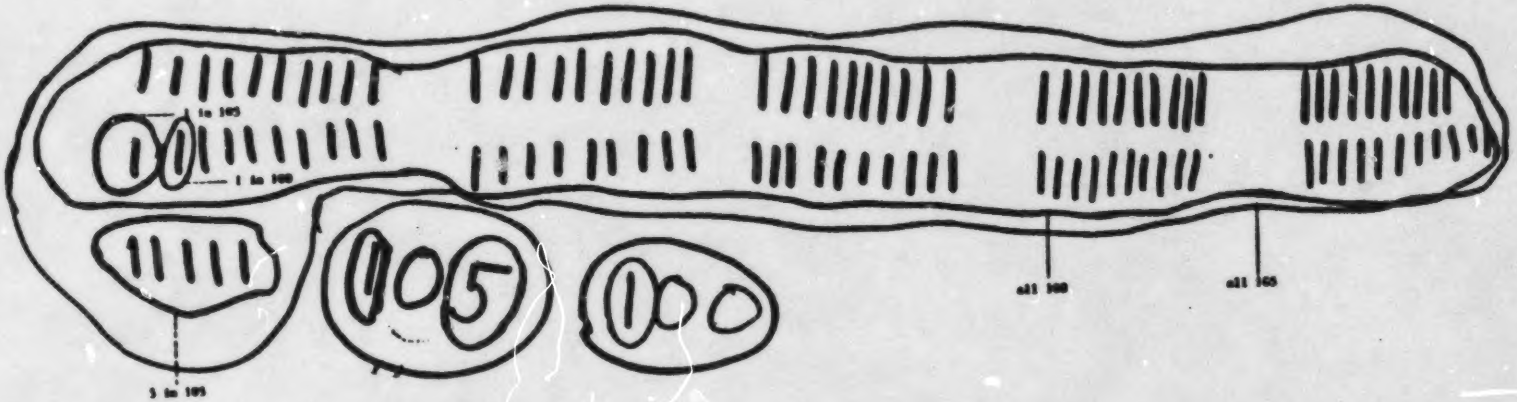
284

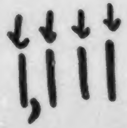

14
cd1 14
1 in 14
4 in 14
20
25
cd1 20
0 in 20
2 in 23
2 in 20
cd1 23
3 in 23


all 20
2 in 20
20
5 in 25
2 in 25
all 25
25
100
105
5 in 105
all 100
all 105
1 in 105
1 in 100

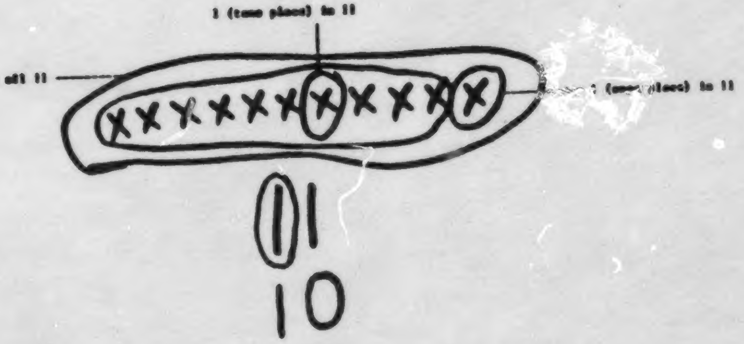
286

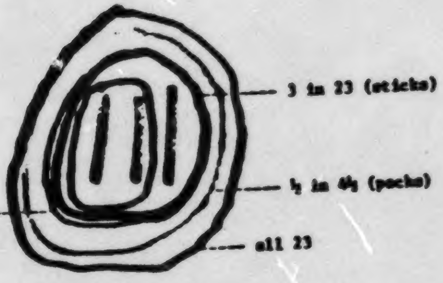
287


1 in 105
1 in 100
5 in 105
105
100
all 100
all 105

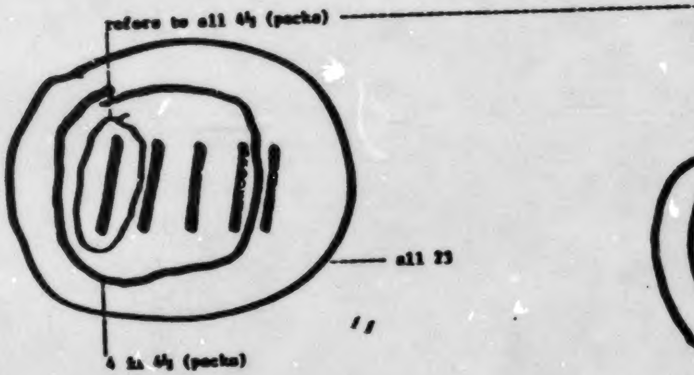


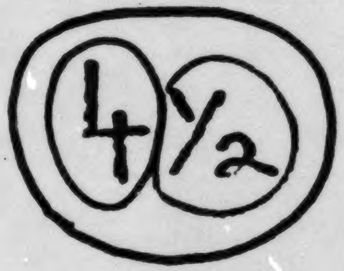
child knew the place names
(ones, tens, hundreds, thousands)

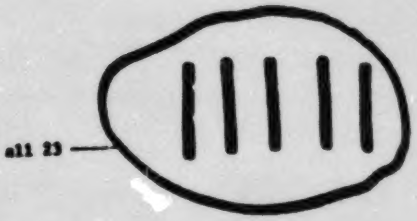

1 (tens place) in 11
all 11
1 (ones place) in 11
01
10

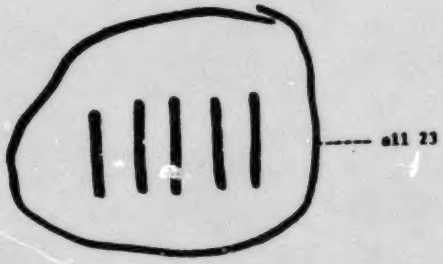

3 in 23 (sticks)
½ in 4½ (packs)
all 23

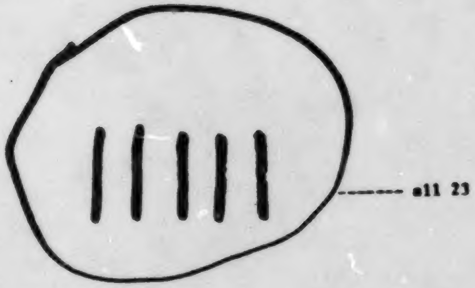
23


refers to all 4½ (packs)
all 23
4 in 4½ (packs)


4½

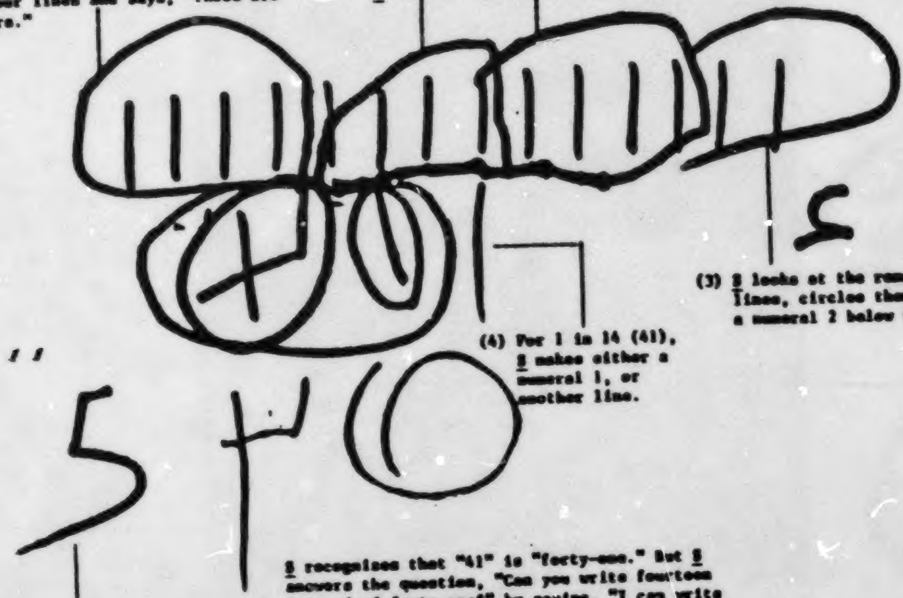

all 23


all 23


all 23

(1) For 4 in 14 (41), subject first circles these four lines and says, "Those are all fours."

(2) "I know how to make another four." S circles two more groups of four.

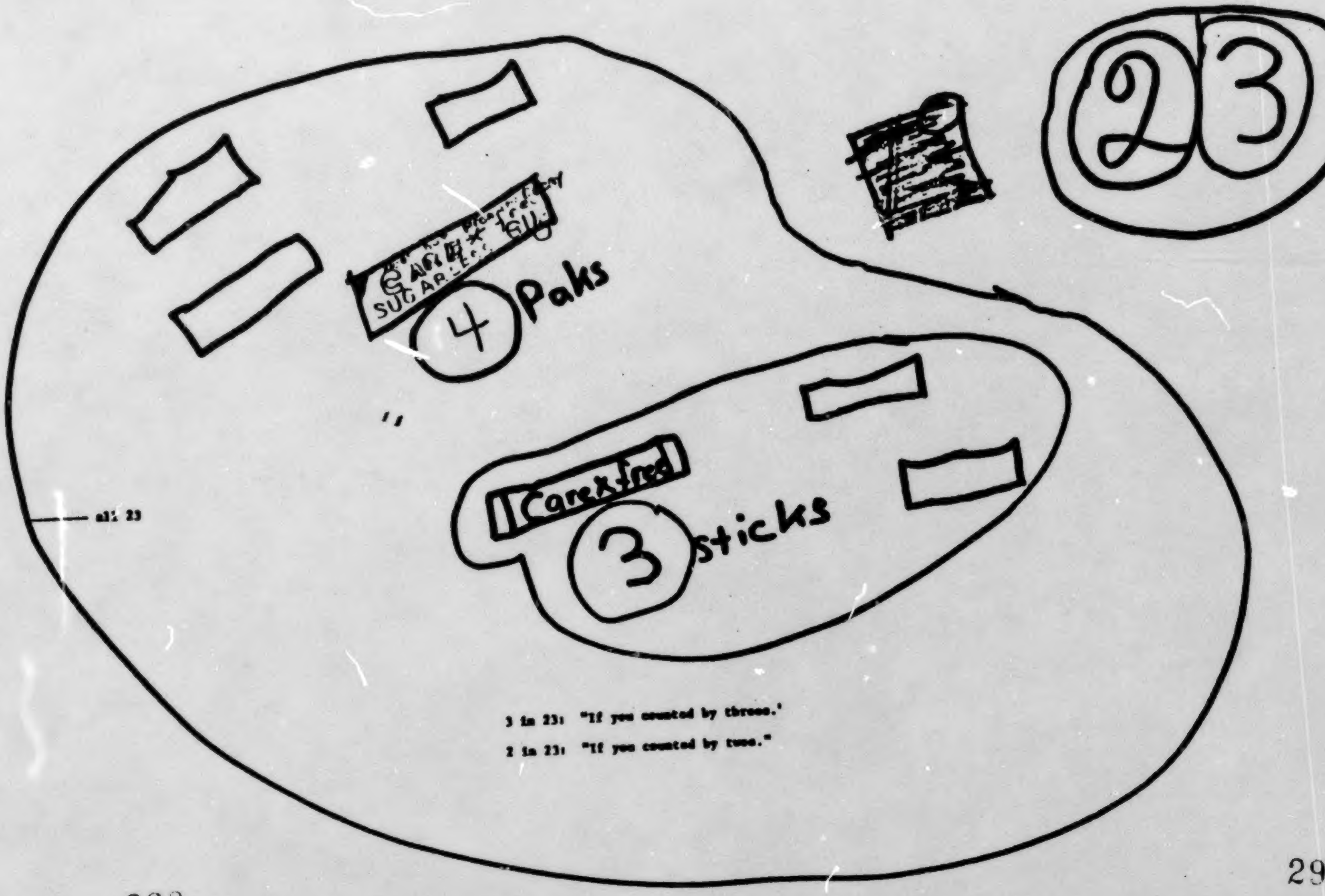


(3) S looks at the remaining two lines, circles them, and writes a numeral 2 below them.

(4) For 1 in 14 (41), S makes either a numeral 1, or another line.

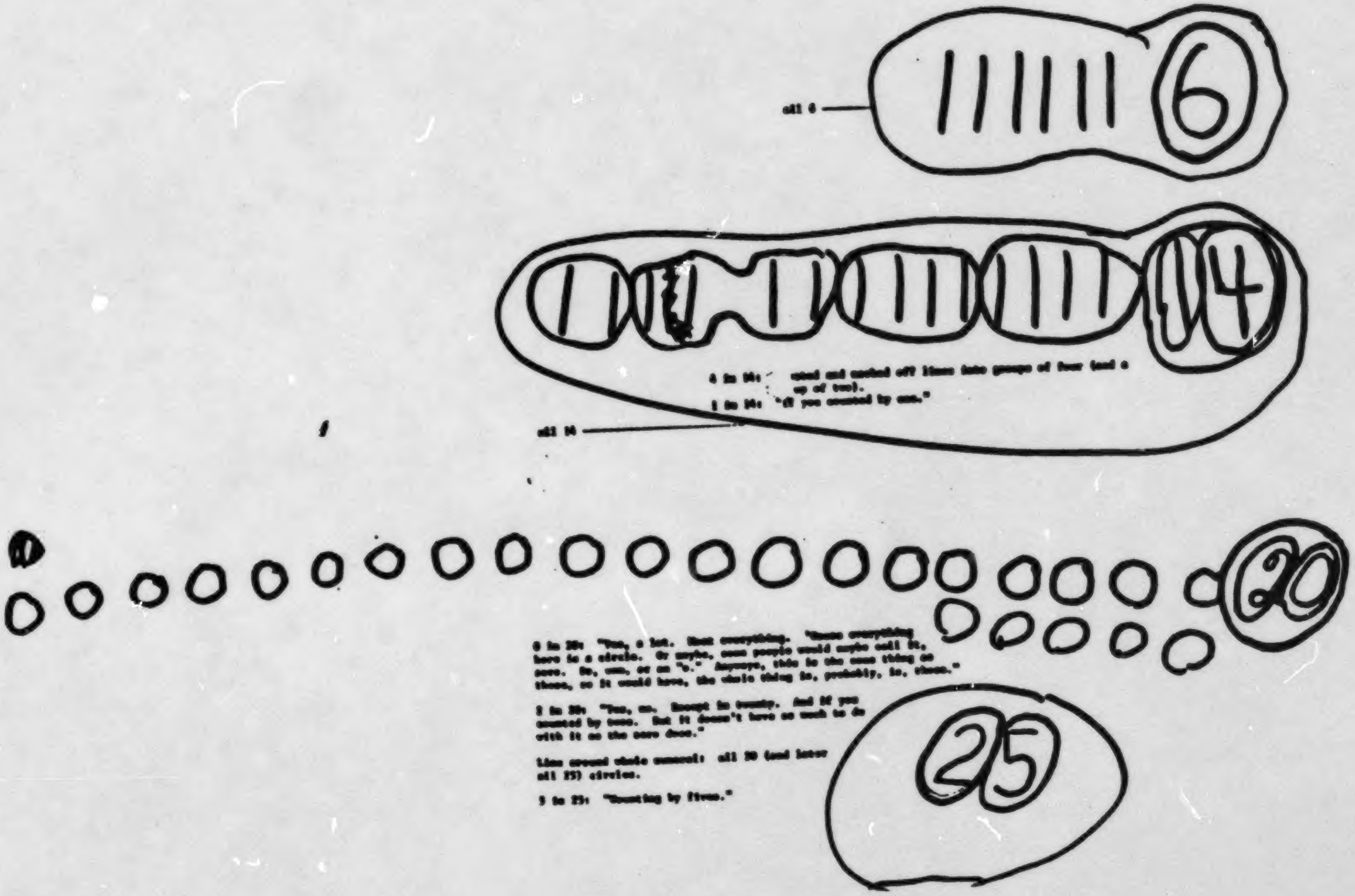
(5) For whole numeral, 14 (41), S says, "That's four, one," and writes the numeral 5 because (4 + 1) = 5.

S recognizes that "41" is "forty-one." But S answers the question, "Can you write fourteen instead of forty-one?" by saying, "I can write forty."


CAREFREE
SUGAR
4 Paks
Carefree
3 sticks
23
all 23

3 in 23: "If you counted by threes."

2 in 23: "If you counted by twos."



6 in 14: uped and marked off lines into groups of four (and a up of two).

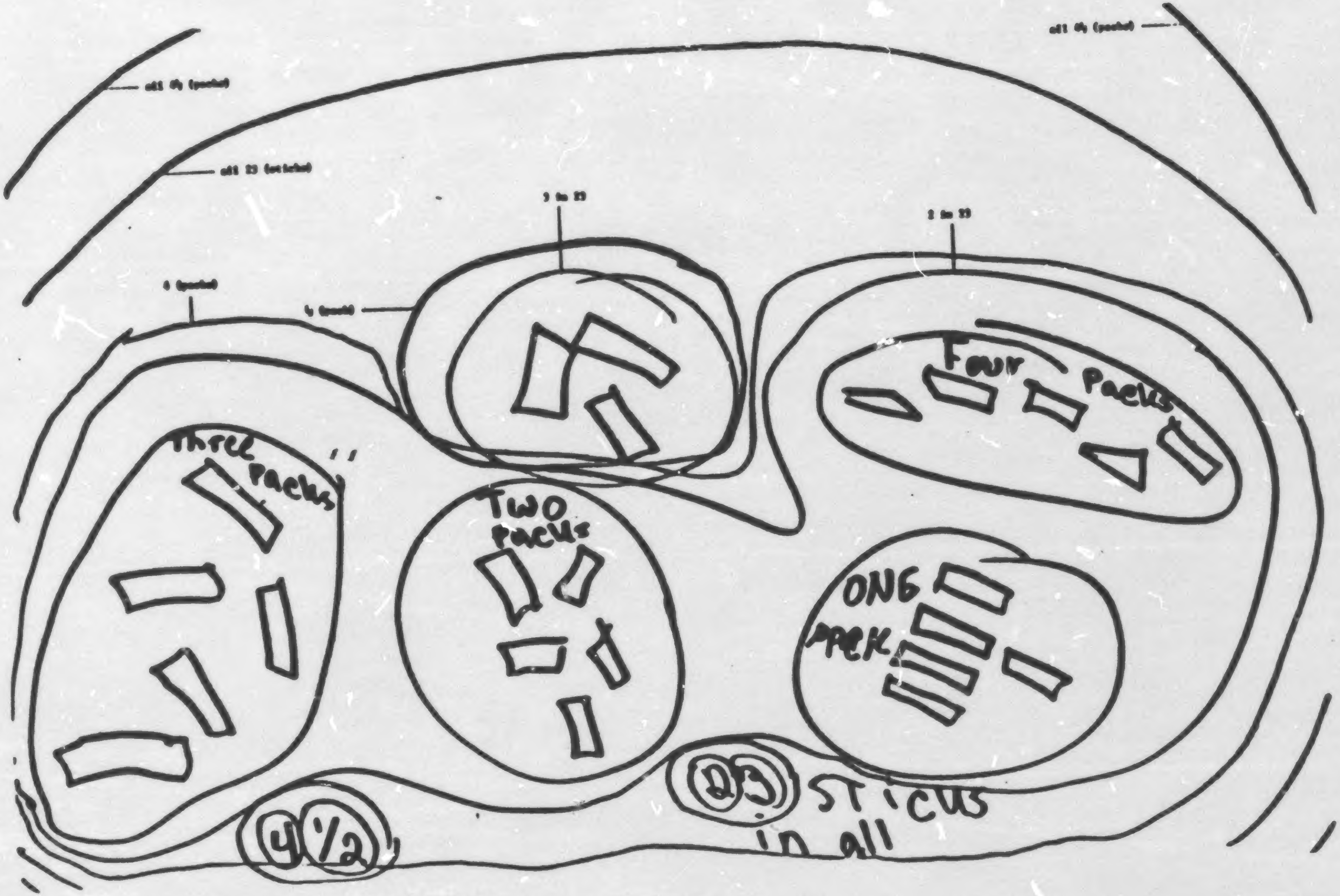
1 in 14: "If you counted by one."

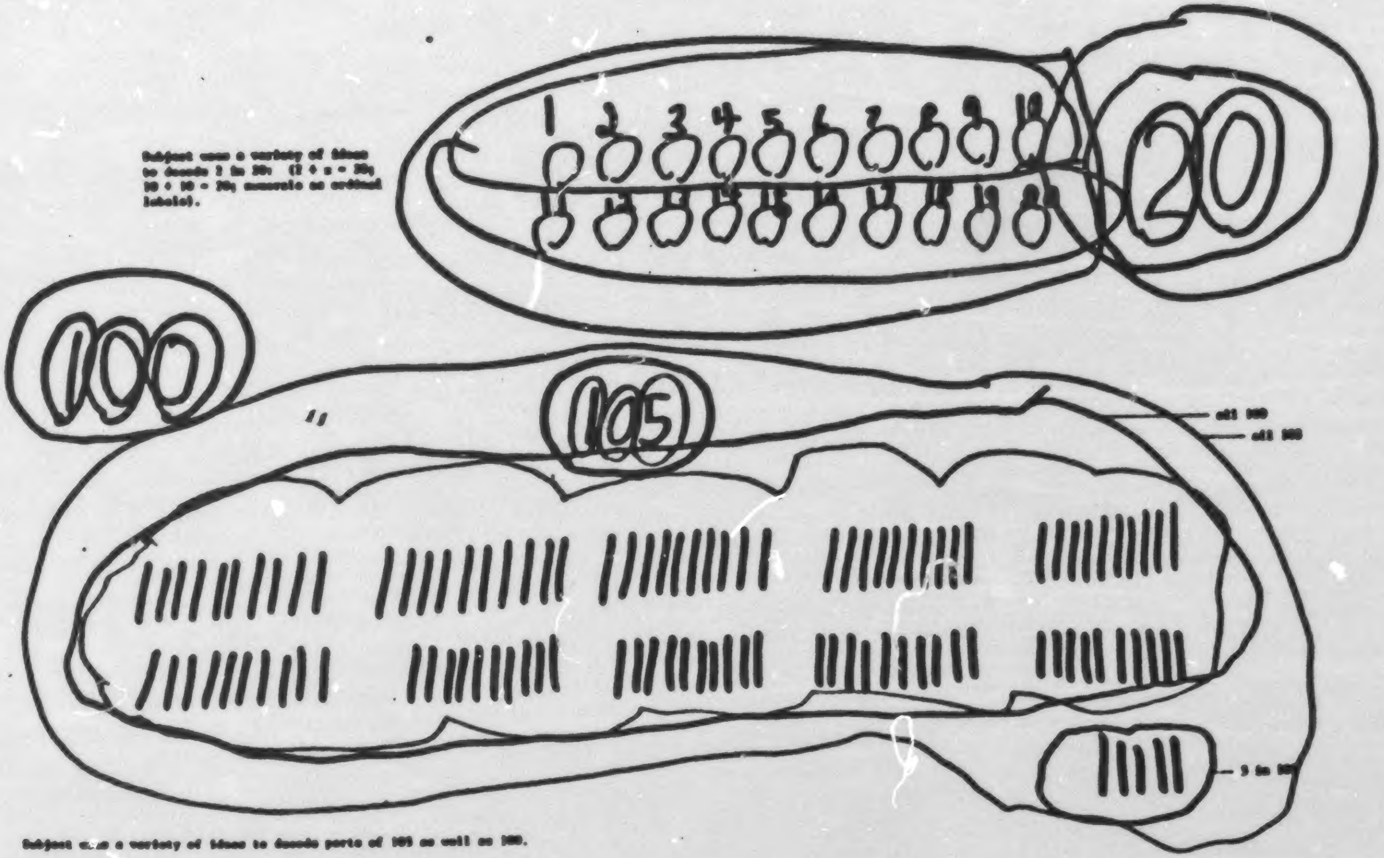
6 in 20: "Yes, a lot. Most everything. 'Cause everything here is a circle. Or maybe, some people would maybe call it, zero. So, um, or an "o." Anyways, this is the same thing as these, so it would have, the whole thing is, probably, is, these."

1 in 20: "Yes, um. Except in twenty. And if you counted by tens. But it doesn't have as much to do with it as the zero does."

Line around whole numeral: all 20 (and later all 25) circles.

5 in 25: "Counting by fives."


all 4s (packs)
all 23 (sticks)
3 in 23
2 in 23
4 (packs)
4 (pack)
all 4s (packs)
Four packs
Three packs
Two packs
One pack
4 1/2
23 sticks in all


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
20
100
105
all 100
all 105
5 in 105

Subject uses a variety of ideas to decode 2 in 20: (2 + x = 20; 10 + 10 = 20; numerals as ordinal labels).

Subject uses a variety of ideas to decode parts of 105 as well as 100.

Appendix B:

Summary of Levels Within Tasks*

1. Conservation of Elementary Number 254

2. Establishing Equality Among Unequal Collections 254

4. Socks and Pairs 255

5. Drawing Six Sticks 255

6. Wheels and Cars

    Grouping Objects in Action 256

    Symbolic Representation 257

    Conventional Representation 258

    Meanings of Digits and Numerals 258

7. Packs of Gum

    Grouping Objects in Action 261

    Symbolic Representation 262

    Conventional Representation 262

8. Other Digits and Numerals 263

9. Marbles 263

---

*Levels were not established for Task 3.

Levels Within Tasks

The ideas and behaviors that seemed to differentiate levels in children's developing knowledge are given below.

Task 1. Conservation of Elementary Number (Piaget, 1941).

Level 1. The child does not establish equivalence between two rows of objects.

Level 2. The child establishes equivalence between two rows, either by one-to-one correspondence or by counting. However the child does not conserve.

Level 3. The child establishes equivalence between the two rows but vacillates between conserving and not conserving number. The child is inconsistent.

Level 4. The child conserves number unequivocally.

Task 2. Establishing Equality Among Unequal Collections.

Level 1. The child uses some non-numerical idea as the basis for making unequal collections "fair."

Example: the child pushes the two collections together and says, "They (the animals) have to share."

Level 2. The child has an intuition that the collections should be the same. But the child's notion is global; he has not differentiated between same numbers (numerical equality), same appearance (spatial arrangement), etc.

Example: the child says the collections have to be the same, but he does not make them equal in number; or he makes the collections equal but cannot explain why the result is "fair."

Level 3. The child thinks that equality can be established in one or two ways: by adding two elements to the smaller collection B (thus A = 6 and B = 6), and/or by removing two elements from the larger collection A (A = 4 and B = 4).

Level 4. The child establishes equality between the collections by moving one element from Collection A to Collection B, thus making two collections of five elements each. The child may do this directly or he may have first made collections of four and/or six objects each.

Task 3. Anticipating Equality Among Unequal Collections Without Counting.

Due to problems in the administration of this task, levels could not be established for it. These problems are elaborated in the discussion of the results, Chapter 5.

Task 4. Socks and Pairs.

Level 1. The child counts the whole collection of individual socks, "one, two, ... six," or "one, two; one, two; one, two." But questions concerning "how many socks vs. how many pairs" are answered with a blank look, or talk about some other topic.

Level 2. The child treats the term "socks" and "pairs" as if they were synonyms. She uses the same number name to identify each.

Example: the child says there are "two, two, two"; "three pairs and three socks," or "six pairs and six socks."

Level 3. The child counts the socks (six), counts the pairs (three), and maintains the idea that there are six socks at the same time as there are three pairs.

Task 5. Drawing Sticks.

Level 1. The child makes some kind of a drawing.

Type A. The child draws something irrelevant to the task, such as a drawing of a house, a person, or an animal.

Type B. The child makes a single drawing of a collection of six sticks. This drawing may or may not show the spatial separation between the sub-collections of sticks.

Level 2. The child makes three separate drawings for the three arrangements of sticks.

Type A. The child shifts the mode of representation within or among the drawings, thus creating a mixture of symbols and signs. For example, collection A is represented with a drawing of four sticks and two sticks with a space separating the sub-collections; collection B is represented with a single numeral, 6; and collection C is represented with two numerals, 5 and 1.

Type B. The child produces drawings in which the sub-collections are ambiguous, and thus it is difficult for him to use the drawings to accurately reproduce the sub-collections.

Note: the child uses one or some combination of the following means to show sub-collections — spatial separation, change of color, and making boundary figures different in size from the rest.

Level 3. The child draws the correct number of sticks (wholes), and the sub-collections (parts) are clearly indicated. However the child compares the sub-collections (parts) within and among the drawings, rather than the wholes. Thus he says that five sticks and four sticks are more than one, two, or three sticks.

Level 4. The child draws each of the collections (wholes) and sub-collections (parts) accurately. He compares the drawings and says that none shows more than any other. "They're all the same." "They all have six."

Task 6. Wheels and Cars.

This task has several parts. Children were asked to (a) group objects in action, (b) symbolically represent their actions, or the results of their actions, (c) write numerals corresponding to the quantities they had drawn, and (d) interpret the meaning of the digits and numerals relative to the quantities they had drawn. Levels for each of the sub-tasks are given separately.

Task 6a. Grouping Objects in Action.

Level 1. The child counts the wheels on the toy car, "one, two, three, four." The collection of twelve wheels, however, remain as separate objects and are not grouped at all.

Example: the child "counts" the total number of wheels (i.e., counts them imprecisely), and initiates "a new game" with the wheels. He makes no effort to group the wheels into sets.

Level 2. The child groups the wheels for one set (car) out of the ungrouped wheels, but leaves the remaining objects ungrouped.

Level 3. The child groups all of the elements into sets, but the set size is wrong.

Note: these children made sets of two, rather than four, wheels. The source of this error seemed to be the child's image of a car from its "side view" where only two wheels are visible. Curiously enough, these children did not give up their idea of "twos" despite the fact that a toy car with four wheels had been used just a few moments before.

Level 4. The child groups all of the objects into numerically correct sets (four wheels per car).

Task 6b. Symbolic Representation.

Level 1. The child represents an objects as such, and not a quantity of objects.

Example: the child draws a car or a truck and ignores the request to draw amounts of wheels.

Level 2. The child draws a quantity of objects.

Type A. The child draws many wheels, or an approximation of the ungrouped numerosity of the whole.

Type B. The child draws one or two sets of wheels.

Level 3. The child represents the numerosity of the whole. In the process of drawing, however, she transforms the objects into something else (e.g., "hamburgers," or "a rabbit and a dog"). Thus she abandons the idea of "groups of four."

Note: the shifting of ideas midstream indicates how fragile the quantitative idea is (a whole of sixteen, and within that whole, sub-collections or sets of four).

Level 2. Single-digit number-squiggles are generally recognized and called by their appropriate name (e.g., "that's the number six"). But rather than making quantitative correspondences between numerals and represented objects, the child seems to use some kind of a "matching schema" to make a link between squiggles and other things. These correspondences are for the most part non-quantitative, though quantitative notions are occasionally mixed in.

Type A. The child makes a correspondence between the colors used in writing squiggles and drawing objects.

Type B. The child makes a verbal (number-name) correspondence between a squiggle and some unrelated instance in which that name is known.

Example: 4 (written to show "how many wheels") elicits, "I know that because I'm four years old," or "I was four before I was five." The connection between the 4 and four wheels, drawn by the child just moments before, is not made.

Type C. The child makes a correspondence between one number-squiggle and any other number-squiggle written on the paper, as if to say "they're both numbers, and therefore they 'match.'"

Type D. The child makes a correspondence between identical number-squiggles. The correspondence is qualitative (identical mark) rather than quantitative (identical mark to signify same amounts).

Level 3. Number-squiggles, and particularly single-digit numerals, can stand for quantities of represented objects. But other ideas operate at the same time, resulting in confusion and inconsistency of responses. The notion that single- and two-digit numerals refer to specific amounts (cardinality) is one among several ideas that are not fully differentiated, one from the other.

Type A. Two-digit numerals cannot be "dissected" into their constituent digits. The number "disappears" when it is broken down into its written parts.

Type B. A whole two-digit numeral, as well as either written part, all refer to the same amount.

Type C. The represented objects can be used to answer one number-question. But the objects, once used, cannot be referred to in order to answer a second number-question.

Level 4. The child represents the numerosity of the whole (sixteen objects) as well as sub-collections or groups of four. The sets are indicated in one or a combination of the following ways — color (a different color for each of the groups of four), spatial grouping (sets of four drawn on different areas of the paper), boundary lines (lines indicating the separation of the whole into groups of four), and labeling (numerals and/or written words to identify the groups).

Task 6c. Conventional Representation (numerals).

Level 1. The child labels individual objects with some kind of mark. For example, she makes a sequence of short lines, one mark underneath each object drawn.

Level 2. The child makes a mark approximating the shape of the number-squiggle. He often makes such remarks as, "A five, that's a backwards two," or "That's how I make a six," as he writes the squiggles.

Examples: Ɛ for 3; Ƨ for 5; ∂ for 6; ‹ for 7.

Level 3. The child makes the conventional mark (appropriate shape) for most single-digit numerals; two-digit numerals are often inverted.

Examples: "61" for 16; "21" for 12.

Level 4. The child writes the conventional marks for all numerals.

Task 6d. Meanings of Digits and Numerals.

Level 1. Number-squiggles are graphic marks that are linked to the objects on which they are found (F. Siegrist and A. Sinclair, research in progress).

Type A. Number-squiggles are "naming labels."

Example: 4 "says car" or 2 is "Channel 2."

Type B. Number-squiggles carry "functional messages."

Example: "they're for things you buy" or 0 (zero) is "for blast-off."

Type C. Number-squiggles have no direct relation to anything written or represented on the paper. The child might circle things because the Interviewer circled things (albeit their number-squiggles).

Type D. Number-squiggles as "ordinal labels" (that is, a sequence of marks identifying separate objects in a sequence of objects) is not differentiated from number-squiggles as signifying "cardinal values."

Example: the 6 in 16 means the sixth wheel; or the whole numeral 16 means the sixteenth wheel.

Type E. In the process of searching for meanings for the separate digits of a two-digit numeral, the "units of meaning" or "referents" change.

Example: the 6 in 16 refers to six wheels, but the 1 in 16 means one car (i.e., six of something and one of something else).

Type F. The operation of addition is applied to the digits making up a two-digit numeral.

Example: the 1 in 16 means one wheel, the 6 in 16 means six wheels, and the whole numeral means "one and six is seven."

Type G. The graphic marks themselves form the focus of meaning.

Example: the child makes a figurative correspondence between the shapes of the numerals (1 is "like a line" and 6 is "like a circle") and other things drawn on the paper. Alternatively, circles drawn around the number-squiggles result in products that "look like a wheel or a machine."

Type H. A numerical correspondence is made between one, but not both, of the written parts of a two-digit numeral and objects.

Example: the 1 in 16 means ten, but the 6 in 16 means nothing at all; or the 6 signifies six objects, but the 1 means nothing.

Level 4. Whole two- and three-digit numerals stand for the totality of the objects represented. The individual digits are consistently transformed into numerals in their own right, and they are treated in one of two ways. In neither case does the child sense a necessary relation between the numerical parts (six objects and ten objects) and the numerical whole (sixteen objects) being represented.

Type A. 1 in 16 signifies one object and 6 in 16 six objects; that nine objects remain unaccounted for is of no concern.

Type B. 1 in 16 stands for sets of one, and 6 in 16 for sets of six objects.

Level 5. The individual digits making up a two- or three-digit numeral stand for amounts that are determined by the place or position in which the digits occur. The mechanisms leading to this understanding of place value consist of a synthesis of three gradually constructed ideas:

(a) Notational rule - 1 in 16 stands for ten because it is written in the tens place.

(b) Numerical part-whole relations - 1 in 16 stands for ten because six and ten add up to sixteen.

(c) Multiplication - 1 in 16 stands for ten because 1 x 10 equals ten.

Task 7. Packs of Gum.

Task 7 is composed of the same sub-tasks as those used in Task 6. However in Task 7, the objects cannot be equally distributed into sets: an uneven number of objects makes it necessary for children to deal with such ideas as "remainders" and "half a pack" (n = 13 for the youngest group of children, and n = 23 for slightly older children).

Task 7a. Grouping Objects in Action.

Level 1. The child counts the gum in the opened pack, "one, two, three, four, five." The collection of thirteen sticks of gum, however, remain as separate objects and are not grouped at all.

Level 2. The child groups the gum for one set (pack) out of the thirteen sticks, but leaves the remaining sticks ungrouped.

Level 3. The child groups the gum for more than one, but not all, of the packs that can be made with the twenty-three sticks of gum.

Level 4. The child groups the sticks of gum into four packs and indicates that there are "not enough" for a fifth pack. Some of these children call the remainders "half a pack."

Task 7b. Symbolic Representation.

Level 1. The child represents an object as such, and not a quantity of objects. For example, the child draws one or two sticks of gum and vacillates between identifying the objects as "sticks" and "packs."

Level 2. The child draws a quantity of objects.

Type A. The child draws many sticks of gum, or an approximation of the ungrouped numerosity of the whole.

Type B. The child draws the sticks for one or two packs of gum.

Level 3. The child represents the numerosity of either the ungrouped whole (all of the sticks, but none of the packs), of the sets (all of the packs but none of the sticks), but not both.

Level 4. The child represents the numerosity of the whole as well as the groups. The child draws either twenty-three sticks clustered into groups of five, with three sticks "left over," or four packs and three sticks.

Task 7c. Conventional Representation (numerals).

Note: please see Task 6c for examples.

Level 1. The child makes some kind of a grphic mark to label individual elements.

Level 2. The child makes a mark approximating the shape of the number-squiggle.

Level 3. The child makes the conventional mark for most single-digit numerals. Two-digit numerals are often inverted.

Level 4. The child writes the conventional marks for all numerals.

Task 7d. Meanings of Digits and Numerals.

Note: please see Task 6d for examples.

Level 1. Number-squiggles are graphic marks that are linked to the objects on which they are found (F. Siegrist and A. Sinclair, research in progress).

Level 2. The child uses "matching schemas" to make non-quantitative correspondences between various graphic marks.

Level 3. The child thinks that number-squiggles, and particularly single-digit numerals, can stand for quantities of represented objects. But this idea is one among several ideas that are not fully differentiated, one from the other.

Level 4. The child thinks that a two- or multi-digit numeral stands for the quantity of objects represented (the whole). The individual digits are consistently transformed into numerals in their own right; they are thought to represent that many objects, or that many sets of objects.

Level 5. The child thinks that the individual digits making up a two-digit numeral stand for amounts that are linked to the position or place in which the digits occur. The child's understanding of place value is facilitated by her grasping of notational rules, numerical part-whole relations, or multiplication.

Task 8. Other Digits and Numerals.

The levels for this task are the same as those for Tasks 6d and 7d.

Task 9. Marbles.

Many children "just wanted to play" and asked the Interviewer to do the score-keeping. Among these children, some were reluctant to try to invent a means of keeping score "without using numbers." But most were getting tired and simply wanted to play. The following is a list of methods employed by the children who did keep score, but they do not constitute "levels." It was interesting to note that no child who used an invented procedure had any difficulty coming up with some means of recording "zero."

Type 1. Tally marks (no necessity for writing anything down for zero).

Type 2. Alphabetic letters (i.e., A = 1, B = 2, C = 3, etc.)

Type 3. Arbitrary "ideographs" (e.g., a "happy face" for 1, an "x" for 2, a slash-mark for 3, etc.).

Type 4. A drawing of a pie, where an appropriate number of pie pieces were colored in for the number of marbles knocked out on each shot.

Appendix C:

Level x Age Analyses for All Tasks

C1. Task 1, Conservation of elementary number x age 265

C2. Task 2, Establishing equality among unequal collections x age 266

C3. Task 4, Socks and pairs x age 267

C4. Task 5, Six sticks x age 268

C5. Task 6a, Grouping wheels x age 269

C6. Task 6b, Symbolizing wheels x age 270

C7. Task 7a, Grouping gum x age 271

C8. Task 7b, Symbolizing gum x age 272

C9. Tasks 6c, 7c and 8, Writing numerals x age 273

C10. Tasks 6d, 7d and 8, Assigning meaning to numerals x age 274

Table C1.
Task 1. Conservation of Elementary Number x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	12	7	4	0	1
5 year olds	15	0	9	1	5
6 year olds	12	0	0	0	12
7 year olds	17	0	0	1	16
8 year olds	12	0	0	0	12
9 year olds	12	0	0	0	12
Collapsed age groups					
4 & 5 yr. olds	27	7	13	1	6
6 & 7 yr. olds	29	0	0	1	28
8 & 9 yr. olds	24	0	0	0	24

$x^2 = 54.34$, $df = 6$

$v = .583$

$p < .001$

Table C2.
Task 2. Establishing Equality Among Unequal Collections x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	12	3	5	2	2
5 year olds	15	0	2	8	5
6 year olds	12	0	0	1	11
7 year olds	17	0	0	0	17
8 year olds	12	0	0	0	12
9 year olds	12	0	0	0	12
Collapsed age groups					
4 & 5 year olds	27	3	7	10	7
6 & 7 year olds	29	0	0	1	28
8 & 9 year olds	24	0	0	0	24

$\chi^2 = 48.47$, df = 6

v = .550

$p < .001$

Table C3.
Task 4. Socks and Pairs x Age

Age	Number of Subjects	Level 1	Level 2	Level 3
4 year olds	11	3	7	1
5 year olds	13	1	10	2
6 year olds	10	0	0	10
7 year olds	16	0	1	15
8 year olds	11	0	0	11
9 year olds	12	0	0	12
Collapsed age groups				
4 & 5 yr. olds	24	4	17	3
6 & 7 yr. olds	26	0	1	25
8 & 9 yr. olds	23	0	0	23

$\chi^2 = 56.02$, df = 4

v = .619

$p < .001$

Table C4.
Task 5. Six Sticks x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	12	7	3	1	1
5 year olds	13	4	4	1	4
6 year olds	9	0	2	3	4
7 year olds	14	1	0	1	12
8 year olds	11	0	1	1	9
9 year olds	12	0	0	0	12
Collapsed age groups					
4 & 5 year olds	25	11	7	2	5
6 & 7 year olds	23	1	2	4	16
8 & 9 year olds	23	0	1	1	21

$\chi^2 = 35.39$, df = 6

$v = .499$

$p < .00$.

Table 65.
Task 6a. Wheels and Cars (Grouping objects in action) x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	12	6	2	0	4
5 year olds	15	2	2	1	10
6 year olds	12	0	1	1	10
7 year olds	17	0	1	0	16
8 year olds	12	0	0	0	12
9 year olds	12	0	0	0	12
Collapsed age groups					
4 & 5 yr. olds	27	8	4	1	14
6 & 7 yr. olds	29	0	2	1	26
8 & 9 yr. olds	24	0	0	0	24

$\chi^2 = 24.51$, df = 6

$v = .391$

$p < .001$

Table C6.
Task 6b. Wheels and Cars (Symbolic Representation) x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	11	7	3	0	1
5 year olds	13	1	5	1	6
6 year olds	12	0	2	2	8
7 year olds	17	0	2	0	15
8 year olds	12	0	0	1	11
9 year olds	12	0	0	1	11
Collapsed age groups					
4 & 5 yr. olds	24	8	8	1	7
6 & 7 yr. olds	29	0	4	2	23
8 & 9 yr. olds	24	0	0	2	22

$x^2 = 34.55$, df = 6

$v = .474$

$p < .001$

Table C7.

Task 7a. Grouping Objects (Gum) in Action x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	8	4	4	0	0
5 year olds	11	4	2	2	3
6 year olds	10	0	0	0	10
7 year olds	17	0	0	1	16
8 year olds	12	0	0	0	12
9 year olds	12	0	0	0	12
Collapsed Age Groups					
4 & 5 year olds	19	8	6	2	3
6 & 7 year olds	27	0	0	1	26
8 & 9 year olds	24	0	0	0	24

$\chi^2 = 52.747$, df = 6

$v = .614$

$p < .001$

Table C8.

Task 7b. Symbolizing Gum x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	8	2	5	1	0
5 year olds	10	1	4	3	2
6 year olds	10	0	3	3	4
7 year olds	16	0	1	2	13
8 year olds	12	0	0	0	12
9 year olds	12	0	0	1	11
Collapsed Age Groups					
4 & 5 year olds	18	3	9	4	2
6 & 7 year olds	26	0	4	5	17
8 & 9 year olds	24	0	0	1	23

$\chi^2 = 36.999$, df = 6

v = .522

$p < .001$

Table C9.
Tasks 6c, 7c, and 8. Writing Numerals x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4
4 year olds	12	4	7	1	0
5 year olds	13	0	6	7	0
6 year olds	12	0	0	7	5
7 year olds	17	0	1	1	15
8 year olds	12	0	0	0	12
9 year olds	12	0	0	0	12
Collapsed age groups					
4 & 5 year olds	25	4	13	8	0
6 & 7 year olds	29	0	1	8	20
8 & 9 year olds	24	0	0	0	24

$\chi^2 = 66.678$, df = 6

v = .671

$p < .001$

Table C10.
Tasks 6d, 7d, and 8. Meanings of Digits and Numerals x Age

Age	Number of Subjects	Level 1	Level 2	Level 3	Level 4	Level 5
4 year olds	12	4	5	3	0	0
5 year olds	13	1	4	8	0	0
6 year olds	11	0	1	4	6	0
7 year olds	16	0	0	5	9	2
8 year olds	11	0	0	0	9	2
9 year olds	12	0	0	1	6	5
Collapsed age groups						
4 & 5 year olds	25	5	9	11	0	0
6 & 7 year olds	27	0	1	9	15	2
8 & 9 year olds	23	0	0	1	15	7

$\chi^2 = 129.26$, df = 8

v = .587

$p < .001$

Appendix D:

Relationships among Levels Within Domains

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D8. Task 4 x Task 6a 283
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D11. Task 5 x Task 6b 286
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D13. Task 6b x Task 7b 288
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Table D1. Cognitive Development: Task 1 x Task 2

Conservation of Elementary Number	Establishing Equality Among Unequal Collections			
	Level 1	Level 2	Level 3	Level 4
Level 1	3	3	1	0
Level 2	0	3	8	2
Level 3	0	1	0	1
Level 4	0	0	2	51

$\chi^2 = 88.02$, df = 9

$v = .625$

$p < .001$

Table D2. Cognitive Development: Task 1 x Task 4

Conservation of Elementary Number	Socks and Pairs Level 1	 Level 2	 Level 3
Level 1	3	4	0
Level 2	0	10	1
Level 3	0	1	1
Level 4	1	3	49

$\chi^2 = 65.36$, df = 6

v = .669

$p < .001$

Table D3. Cognitive Development: Task 1 x Task 6a

	Grouping Objects (Wheels) in Action			
Conservation of Elementary Number	Level 1	Level 2	Level 3	Level 4
Level 1	6	0	0	1
Level 2	2	3	0	8
Level 3	0	0	0	2
Level 4	0	3	2	42

$\chi^2 = 49.16$, df = 9

$v = .487$

$p < .001$

Table D4. Cognitive Development: Task 1 x Task 7a

	Grouping Objects (Gum) in Action			
Conservation of Elementary Number	Level 1	Level 2	Level 3	Level 4
Level 1	3	1	0	0
Level 2	5	3	2	0
Level 3	0	0	0	1
Level 4	0	2	1	52

$\chi^2 = 61.839$, df = 9

$v = .543$

$p < .001$

Table D3. Cognitive Development: Task 2 x Task 4

	Socks and Pairs		
Establishing Equality Among Unequal Collections	Level 1	Level 2	Level 3
Level 1	2	1	0
Level 2	1	5	0
Level 3	0	9	1
Level 4	1	2	51

$\chi^2 = 77.5$, df = 6

$v = .729$

$p < .001$

Table D6. Cognitive Development: Task 2 x Task 6a

Establishing Equality Among Unequal Collections	Grouping Objects (Wheels) in Action: Level 1	Level 2	Level 3	Level 4
Level 1	3	0	0	0
Level 2	3	1	0	3
Level 3	2	2	0	7
Level 4	0	3	2	54

$\chi^2 = 46.96$, df = 9

$v = .442$

$p < .001$

Table D7. Cognitive Development: Task 2 x Task 7a

Establishing Equality Among Unequal Collections	Grouping Objects (Gum) in Action Level 1	Level 2	Level 3	Level 4
Level 1	1	0	0	0
Level 2	3	1	0	0
Level 3	3	3	1	1
Level 4	1	2	2	52

$\chi^2 = 52.669$, df = 9

v = .501

$p < .001$

Table D8. Cognitive Development: Task 4 x Task 6a

	Grouping Objects (Wheels) in Action			
Socks and Pairs	Level 1	Level 2	Level 3	Level 4
Level 1	3	0	0	1
Level 2	5	3	0	9
Level 3	0	2	2	48

$\chi^2 = 32.67$, df = 6

$v = .473$

$p < .001$

Table D9. Cognitive Development: Task 4 x Task 7a

	Grouping Objects (Gum) in Action			
Socks and Pairs	Level 1	Level 2	Level 3	Level 4
Level 1	1	1	0	0
Level 2	7	4	1	2
Level 3	0	1	1	50

$\chi^2 = 51.163$, df = 6

$v = .613$

$p < .001$

Table D10. Cognitive Development: Task 6a x Task 7a

	Grouping Objects (Gum) in Action			
Grouping Objects (Wheels) in Action	Level 1	Level 2	Level 3	Level 4
Level 1	3	2	0	0
Level 2	1	1	0	2
Level 3	0	0	0	2
Level 4	4	3	3	49

$\chi^2 = 25.610$, df = 9

$v = .349$

$p < .005$

Table D11. Symbolic Representation: Task 5 x Task 6b

	Symbolizing Objects (Wheels)			
Six Sticks	Level 1	Level 2	Level 3	Level 4
Level 1	4	6	0	2
Level 2	1	2	1	5
Level 3	1	1	0	4
Level 4	1	2	4	35

$\chi^2 = 29.57$, df = 9

v = .409

$p < .001$

Table D12. Symbolic Representation: Task 5 x Task 7b

	Symbolizing Gum			
Six Sticks	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Level 1	2	5	1	1
Level 2	0	3	3	1
Level 3	1	1	1	3
Level 4	0	2	4	33

$\chi^2 = 36.470$, df = 9

v = .446

$p < .001$

Table D13. Symbolic Representation: Task 6b x Task 7b

	Symbolizing Gum			
Symbolizing Wheels	Level 1	Level 2	Level 3	Level 4
Level 1	2	1	0	0
Level 2	1	7	0	1
Level 3	0	2	1	2
Level 4	0	3	10	39

$\chi^2 = 62.922$, df = 9

v = .551

$p < .001$

Table D14. Conventional Representation: The Relationship Between Writing Numerals and Assigning Meaning to Numerals

	Meanings of Digits and Numerals				
Writing Numerals	Level 1	Level 2	Level 3	Level 4	Level 5
Level 1	4	0	0	0	0
Level 2	1	8	1	0	0
Level 3	0	3	11	3	0
Level 4	0	0	6	22	9

$\chi^2 = 115.1$, df = 12

v = .751

$p < .001$

Appendix E:

Relationships Among Levels Between Domains

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E29. Task 5 x Tasks 6d, 7d and 8 320

E30. Task 6b x Tasks 6d, 7d and 8 321

E31. Task 7b x Tasks 6d, 7d and 8 322

Table E1. Relationships Between Levels in Cognitive and Symbolic Representation: Task 1 x Task 5

	Six Sticks			
Conservation of Elementary Number	Level 1	Level 2	Level 3	Level 4
Level 1	5	1	1	0
Level 2	4	6	0	2
Level 3	0	0	1	1
Level 4	2	3	5	39

$\chi^2 = 49.67$, df = 9

$v = .486$

$p \leq .001$

Table E2. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 1 x Task 6b

Conservation of Elementary Number	Symbolizing Objects (Wheels)			
	Level 1	Level 2	Level 3	Level 4
Level 1	6	1	0	0
Level 2	0	7	1	4
Level 3	0	0	0	1
Level 4	1	4	5	41

$\chi^2 = 69.486$, df = 9

v = .571

$p < .001$

Table E3. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 1 x Task 7b

Conservation of Elementary Number	Symbolizing Gum Level 1	Level 2	Level 3	Level 4
Level 1	2	2	0	0
Level 2	0	7	2	0
Level 3	0	0	0	1
Level 4	1	4	7	42

$x^2 = 74.467$, df = 9

v = .604

$p < .001$

Table E4. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 2 x Task 5

	Six Sticks			
Establishing Equality Among Unequal Collections	Level 1	Level 2	Level 3	Level 4
Level 1	2	0	1	0
Level 2	2	4	1	0
Level 3	5	2	0	3
Level 4	3	4	5	39

$\chi^2 = 39.141$, df = 9

$v = .429$

$p < .001$

Table E5. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 2 x Task 6b

Establishing Equality Among Unequal Collections	Symbolizing Wheels Level 1	Level 2	Level 3	Level 4
Level 1	3	0	0	0
Level 2	3	2	1	0
Level 3	0	5	1	4
Level 4	1	5	4	49

$\chi^2 = 64.907$, df = 9

v = .527

$p < .001$

Table B6. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 2 x Task 7b

Establishing Equality Among Unequal Collections	Symbolizing Gum Level 1	Level 2	Level 3	Level 4
Level 1	1	0	0	0
Level 2	1	2	0	0
Level 3	0	6	1	1
Level 4	1	5	9	41

$\chi^2 = 54.963$, df = 9

$v = .519$

$p < .001$

Table B7. **Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 4 x Task 5**

	Six Sticks			
Socks and Pairs	Level 1	Level 2	Level 3	Level 4
Level 1	4	0	0	0
Level 2	6	5	2	5
Level 3	2	4	5	37

$\chi^2 = 36.161$, df = 6

v = .508

$p < .001$

Table E8. Relationships Between Levels in Cognitive and Symbolic Representation Tasks:
Task 4 x Task 6b

	Symbolizing Objects (Wheels)			
Socks and Pairs	Level 1	Level 2	Level 3	Level 4
Level 1	3	1	0	0
Level 2	4	7	1	5
Level 3	0	3	5	46

$\chi^2 = 47.995$, df = 6

v = .566

$p < .001$

Table B9. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 4 x Task 7b

	Symbolizing Gum			
Socks and Pairs	Level 1	Level 2	Level 3	Level 4
Level 1	2	0	0	0
Level 2	1	9	1	2
Level 3	0	4	8	40

$\chi^2 = 72.393$, df = 6

v = .735

$p < .001$

Table B10. Relationships Between Levels in Cognitive and Symbolic Representation Tasks:
Task 6a x Task 5

	Six Sticks			
Grouping Objects (Wheels) in Action	Level 1	Level 2	Level 3	Level 4
Level 1	4	3	1	0
Level 2	3	2	0	1
Level 3	0	0	0	1
Level 4	3	4	6	39

$\chi^2 = 30.766$, df = 9

v = .391

$p < .001$

Table E11. Relationships Between Levels in Cognitive and Symbolic Representation Tasks:
Task 6 a x Task 6b

	Symbolizing Objects (Wheels)			
Grouping Objects (Wheels) in Action	Level 1	Level 2	Level 3	Level 4
Level 1	5	2	0	1
Level 2	1	5	0	0
Level 3	0	0	2	0
Level 4	1	5	4	52

$\chi^2 = 85.536$, df = 9

v = .605

$p < .001$

Table E12. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 6a x Task 7b

	Symbolizing Gum			
Grouping Objects (Wheels) in Action	Level 1	Level 2	Level 3	Level 4
Level 1	2	2	0	0
Level 2	0	3	0	1
Level 3	0	1	0	1
Level 4	1	7	10	40

$x^2 = 36.694$, df = 9

$v = .424$

$p < .001$

Table E13. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 7a x Task 5

Grouping Objects (Gum) in Action	Six Sticks Level 1	Level 2	Level 3	Level 4
Level 1	4	1	1	2
Level 2	4	2	0	0
Level 3	0	2	0	0
Level 4	1	3	5	38

$x^2 = 49.382$, df = 9

$v = .511$

$p < .001$

Table E14. Relationships Between Levels in Cognitive and Symbolic Representation Tasks: Task 7a x Task 6b

	Symbolizing Wheels			
Grouping Objects (Gum) in Action	Level 2	Level 2	Level 3	Level 4
Level 1	3	4	0	1
Level 2	0	3	1	2
Level 3	0	0	0	3
Level 4	0	3	4	46

$\chi^2 = 47.347$, df = 9

v = .475

$p < .001$

Table E15. Relationships Between Levels in Cognitive and Symbolic Representation Tasks:
Task 7a x Task 7b

Grouping Objects (Gum) in Action	Symbolizing Gum Level 1	Level 2	Level 3	Level 4
Level 1	2	5	0	0
Level 2	1	4	1	0
Level 3	0	1	2	0
Level 4	0	3	7	42

$\chi^2 = 55.540$, df = 9

v = .522

$p < .001$

Table B16. Relationships Between Levels in Cognitive and Conventional Representation Tasks (Writing Numerals): Task 1 x Tasks 6c, 7c and 8

	Writing Numerals			
Conservation of Elementary Number	Level 1	Level 2	Level 3	Level 4
Level 1	4	3	0	0
Level 2 4	0	9	3	0
Level 3	0	0	0	1
Level 4	0	2	13	38

$\chi^2 = 81.848$, df = 9

$v = .611$

$p < .001$

Table E17. Relationships Between Levels in Cognitive and Conventional Representation Tasks (Writing Numerals): Task 2 x Tasks 6c, 7c, and 8

Establishing Equality Among Unequal Collections	Writing Numerals			
	Level 1	Level 2	Level 3	Level 4
Level 1	3	0	0	0
Level 2	1	5	0	0
Level 3	0	6	4	0
Level 4	0	3	12	44

$\chi^2 = 105.739$, df = 9

$v = .672$

$p < .001$

Table E18. Relationships Between Levels in Cognitive and Conventional Representation Tasks (Writing Numerals): Task 4 x Tasks 6c, 7c, and 8

	Writing Numerals			
Socks and Pairs	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Level 1	3	1	0	0
Level 2	1	11	4	1
Level 3	0	1	11	41

$\chi^2 = 82.651$, df = 6

v = .747

$p < .001$

Table E19. Relationships Between Levels in Cognitive and Conventional Representation Tasks (Writing Numerals): Task 6a x Tasks 6c, 7c, and 8

Grouping Objects (Wheels) in Action	Writing Numerals			
	Level 1	Level 2	Level 3	Level 4
Level 1	4	3	1	0
Level 2	0	3	1	2
Level 3	0	0	2	0
Level 4	0	8	12	42

$\chi^2 = 55.489$, df = 9

v = .487

$p < .001$

Table E20. Relationships Between Levels in Cognitive and Conventional Representation Tasks (Writing Numerals): Task 7a x Tasks 6c, 7c and 8

Grouping Objects (Gum) in Action	Writing Numerals			
	Level 1	Level 2	Level 3	Level 4
Level 1	2	6	0	0
Level 2	0	4	2	0
Level 3	0	0	2	1
Level 4	0	1	9	43

$x^2 = 68.257$, df = 9

$v = .570$

$p < .001$

Table E21. Relationships Between Levels in Cognitive and Conventional Representation (Assigning Meaning to Numerals): Task 1 x Tasks 6d, 7d, and 8

	Meanings of Digits and Numerals				
Conservation of Elementary Number	Level 1	Level 2	Level 3	Level 4	Level 5
Level 1	4	2	1	0	0
Level 2	0	7	5	0	0
Level 3	0	0	1	0	0
Level 4	1	1	14	30	9

$\chi^2 = 72.037$. df = 12

v = .5658

$p < .001$

Table E22. Relationship Between Levels in Cognitive and Conventional Representation (Assigning Meaning to Numerals): Task 2 x Tasks 6d, 7d, and 8

Establishing Equality Among Unequal Collections	Meaning of Digits and Numerals Level 1	Level 2	Level 3	Level 4	Level 5
Level 1	3	0	0	0	0
Level 2	1	4	1	1	0
Level 3	0	5	5	0	0
Level 4	1	0	14	30	9

$\chi^2 = 87.81$, df = 12

v = .6289

$p < .001$

Table E23. Relationships Between Levels in Cognitive and Conventional Representation (Assigning Meaning to Numerals): Task 4 x Tasks 6d, 7d, and 8

	Meanings of Digits and Numerals				
Socks and Pairs	Level 1	Level 2	Level 3	Level 4	Level 5
Level 1	4	0	0	0	0
Level 2	1	8	7	1	0
Level 3	0	1	11	29	9

$\chi^2 = 90.13$, df = 8

$v = .7967$

$p < .001$

Table E26. Relationships Between Levels in Cognitive and Conventional Representation (Assigning Meaning to Numerals): Task 6a x Tasks 6d, 7d, and 8

	Meanings of Digits and Numerals				
Grouping Objects (Wheels) in Action	Level 1	Level 2	Level 3	Level 4	Level 5
Level 1	4	3	1	0	0
Level 2	0	2	2	1	0
Level 3	0	0	1	0	0
Level 4	1	5	16	30	9

$\chi^2 = 42.959$, df $= 12$

$v = .437$

$p < .001$

Table E25. Relationships Between Levels in Cognitive and Conventional Representation Tasks (Assigning Meaning to Numerals): Task 7a x Tasks 6d, 7d and 8

Grouping Objects (Gum) in Action	Assigning Meaning to Numerals Level 1	Level 2	Level 3	Level 4	Level 5
Level 1	2	3	3	0	0
Level 2	1	3	2	0	0
Level 3	0	0	2	1	0
Level 4	0	1	10	29	9

$\chi^2 = 44.280$, df = 12

v = .473

$p < .001$

Table E26. Relationship Between Levels in Symbolic and Conventional Representation Tasks (Writing Numerals): Task 5 x Tasks 6c, 7c and 8

	Writing Numerals			
Six Sticks	Level 1	Level 2	Level 3	Level 4
Level 1	3	7	1	1
Level 2	0	4	4	2
Level 3	1	0	2	3
Level 4	0	2	6	33

$x^2 = 45.131$, df = 9

v = .467

$p < .001$

Table E27. Relationships Between Levels in Symbolic and Conventional Representation (Writing Numerals): Task 6b x Tasks 6c, 7c and 8

	Writing Numerals			
Symbolizing Wheels	Level 1	Level 2	Level 3	Level 4
Level 1	4	2	1	0
Level 2	0	9	1	2
Level 3	0	1	3	2
Level 4	0	2	11	40

$x^2 = 83.545$, df = 9

v = .598

$p < .001$

Table E28. Relationships Between Levels in Symbolic and Conventional Representation Tasks (Writing Numerals): Task 7b x Tasks 6c, 7c, and 8

Symbolizing Gum	Writing Numerals Level 1	Level 2	Level 3	Level 4
Level 1	2	1	0	0
Level 2	0	8	3	2
Level 3	0	0	8	2
Level 4	0	0	3	38

$\chi^2 = 109.102$, df = 9

v = .737

$p < .001$

Table E29. Relationships Between Levels in Symbolic and Conventional Representation (Assigning Meaning to Numerals): Task 5 x Tasks 6d, 7d, and 8

	Meanings of Digits and Numerals				
Six Sticks	**Level 1**	**Level 2**	**Level 3**	**Level 4**	**Level 5**
Level 1	4	4	3	0	0
Level 2	0	4	3	3	0
Level 3	1	1	0	3	1
Level 4	0	1	11	21	8

$\chi^2 = 40.98$, df = 12

v = .448

$p < .001$

Table E30. Relationships Between Levels in Symbolic and Conventional Representation (Assigning Meaning to Numerals): Task 6b x Tasks 6d, 7d, and 8

	Meaning of Digits and Numerals				
Symbolizing Wheels	Level 1	Level 2	Level 3	Level 4	Level 5
Level 1	4	1	2	0	0
Level 2	1	6	2	1	0
Level 3	0	1	3	1	0
Level 4	0	1	13	28	9

$\chi^2 = 67.20$, df $= 12$

$v = .554$

$p < .001$

Table E31. Relationships Between Levels in Symbolic and Conventional Representation (Assigning Meaning to Numerals): Task 7b x Tasks 6d, 7d, and 8

	Assigning Meaning to Numerals				
Symbolizing Gum	**Level 1**	**Level 2**	**Level 3**	**Level 4**	**Level 5**
Level 1	3	0	0	0	0
Level 2	0	6	4	2	0
Level 3	0	0	8	2	0
Level 4	0	0	5	26	9

$\chi^2 = 117.289$, df = 12

$v = .776$

$p < .001$

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